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FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

PROBLEMS IN NONLINEAR OPTICS
(SELECTED CHAPTERS)

by

S. A. Akhmanov and R. V. Khokhlov

FOREIGN TECHNOLOGY DIVISION

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PROBLEMS IN NONLINEAR OPTICS (SELECTED CHAPTERS)

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

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А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

C H A P T E R II

BASES OF THE THEORY OF WAVES IN A NONLINEAR DISPERSIVE MEDIUM

§ 1. Introduction

In the investigation of wave processes in a nonlinear medium, the initial system is the system (I. of I) in which the bond between vectors D and E (material equations) is nonlinear. A general solution of the thus obtained nonlinear system of equations is impossible.

At the same time, for the majority of practically interesting cases, it is possible to develop an effective method of obtaining the approximate solutions based on the circumstance that linear losses in the medium and nonlinear part of the vector of polarization can usually be considered small [see Introduction, formulas (I.14)-(I.16)]. We will subsequently denote small values by the dimensionless parameter $\mu (\mu \ll 1)$. Here we will consider the linear losses by the magnitude of the first order of smallness with respect to μ, so that

$$\hat{\varkappa}(\omega) = Re\,\hat{\varkappa}(\omega) + i\mu\, Im\,\hat{\varkappa}(\omega). \quad (2.1)$$

Nonlinear terms in the decomposition of vector of polarization P with respect to E will ascribe the first and higher order of smallness with respect to μ. It is natural to consider that in the quadratic medium tensors $\chi \sim \mu$ and $\theta \sim \mu^2$ corresponding to the dipole radiation and tensors of higher ranks have an order of μ^3 etc. The lowest nonlinear term is obviously the largest. In the cubic medium the

tensor $\hat{\theta}$ is the lowest, and therefore here we will consider $\theta \sim \mu$. Equations (I.1) can be converted to one second order equation, which in accordance with that mentioned above about the order of smallness of nonlinear and dissipative terms will be recorded in the form:

$$[\nabla[\nabla\mathbf{E}]] + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}^{(n)'}}{\partial t^2} + \mu F\left(t, \mathbf{r}, \mathbf{E}, \frac{\partial^2\mathbf{E}}{\partial t^2}\right) = 0, \quad (2.2)$$

where function F includes components of linear polarization connected with the losses and terms determined by the nonlinear polarization of the medium; the part of the vector of linear polarization determined by Re $\hat{\chi}$ is designated by $P^{(n)1}$.

In a zero approximation ($\mu = 0$) equation (2.2) describes the linear nondissipative medium. Natural waves of such a medium are monochromatic plane waves of constant amplitude; directions of polarizations and wave vectors of natural waves are determined by the properties (2.2). For small μ it is natural to assume that waves in a nonlinear dissipative medium differ little from natural waves of a linear transparent medium. Therefore, if for $\mu = 0$

$$\mathbf{E}(t, \mathbf{r}) = \sum_n \mathbf{A}_{0n} e^{i(\omega_n t - \mathbf{k}_n \mathbf{r})} \quad (2.3)$$

and A_{0n} – constant complex numbers, then for $\mu \neq 0$

$$\mathbf{E}(t, \mathbf{r}) = \sum_n \mathbf{A}_n(\mu\mathbf{r}) e^{i(\omega_n t - \mathbf{k}_n \mathbf{r})}, \quad (2.4)$$

where complex amplitudes are slowly changing functions of the radius vector r. Thus, the dependence field strength of the wave in the nonlinear dissipative medium on r enters in two ways:

a) through the exponential in (2.4). Here coordinate r is the "rapid" spatial scale of changes of the field, which are connected with the "rapid" coordinate, and has an order of $z_б = \frac{2\pi}{k} = \lambda$;

b) through the complex amplitude A_n. Here coordinate r is "slow," which is noted by factor μ. The spatial scale of changes of the field, characterized by the "slow" coordinate, has an order of $\tilde{z}_м \approx \frac{\lambda}{\mu}$; relative changes of complex amplitudes in a weakly

nonlinear, weakly absorbing medium on the wavelength are small $\frac{\Delta_\lambda A_n}{A_n} \ll 1$.

Although the method of designing of approximate solutions of the type (2.4) proves to be basically similar to the corresponding method, developed in the theory of nonlinear oscillations of systems with concentrated constants (see, for example, [53] and [54]), it is expedient before passing to nonlinear problems to illustrate it at first using the simplest example of the linear dissipative medium. It is necessary to stress that in force what was said in the introduction and in Chapter I, the greatest interest for the examined range of questions is in problems on the propagation of nonlinear waves in anisotropic dispersive media. In optics only for an anisotropic (uniform or nonuniform) medium is the obtaining of considerable ratios $\frac{l_{kn}}{\lambda}$ [1] possible. Therefore, in this chapter the following order of consideration of problems about nonlinear waves is accepted. At first certain relationships characterizing natural waves in an anisotropic nonabsorbing medium ($\mu = 0$) are deduced. Then, in the example of linear dissipative anisotropic medium a generalization of the method of slowly changing amplitudes on distributed systems is given; here and subsequently we are limited, as a rule, by the first approximation, i.e., by the constructing of solutions satisfying the initial equation (2.2) to within terms $\sim\mu^2$. Finally, the method of slowly changing amplitudes is used for consideration of a number of model nonlinear interactions in the quadratic and cubic medium. Here one should stress, however, that although the thus obtained system of truncated equations of the first order is considerably simpler than the initial equation, it, for majority of cases does not allow an exact analytic solution. Therefore, in §§ 5-6 of this chapter certain possibilities of further simplification of the problem, already in the stage of consideration of truncated equations, are examined.

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[1]See also Chapter III, § 2, where specific dispersion characteristics of a number of crystals utilized in nonlinear optics are examined.

§ 2. Waves in a Linear Anisotropic Dispersive Medium

2.1. Zero Approximation ($\mu = 0$). Natural Waves of an Anisotropic Nonabsorbing Medium

The process of propagation of waves in an anisotropic linear dispersive medium is described by the wave equation:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + 4\pi \frac{\partial^2 \mathbf{P}^{(n)}}{\partial t^2} + c^2 [\nabla [\nabla \mathbf{E}]] = 0. \quad (2.5)$$

The vector of polarization $P^{(n)}$ is connected with field E by the linear functional relationship:

$$\mathbf{P}^{(n)} = \int_0^\infty \hat{\varkappa}(t') \mathbf{E}(t - t')\, dt', \quad (2.6)$$

where $\hat{\varkappa}(t')$ in the examined case is a tensor with components κ_{mn}.

Let us consider subsequently certain necessary relationships for plane harmonic waves of constant amplitude:

$$\mathbf{E} = \mathbf{e} A_0 e^{i(\omega t - \mathbf{kr})}, \quad (2.7)$$

where A_0 – constant and e – unit vector. The connection between the wave vector k and frequency ω can be obtained if one were to substitute (2.7) into (2.5). It has the form:

$$\omega^2 \mathbf{e} + 4\pi\omega^2 \hat{\varkappa}(\omega) \mathbf{e} + \\ + c^2 [\mathbf{k}[\mathbf{ke}]] = 0, \quad (2.8)$$

where $\hat{\varkappa}(\omega)$ spectral form $\hat{\varkappa}(t')$ (see (1.8)). From a consideration of (2.8) as systems of equations for components of vector e there follows the condition of compatibility of the system – equality to zero of the determinant composed of the coefficients with components of vector e. This relationship, being one of the fundamental equations of crystal optics, gives at the assigned direction k and assigned tensor $\hat{\varkappa}$ two values of the modulus of the wave vector $|k; k_2|$. We assume that they are not equal each other. Each of these values of k corresponds to its own system of components e, i.e., the

assigned polarization of the "natural" wave. Let us designate the unit vectors in the direction of the "natural polarizations" by e_1 and e_2. Vectors $[k_1e_1]$ and $[k_2e_2]$, which have directions of intensities of magnetic fields $H_{1,2}$, are mutually perpendicular, whereas the very e_1 and e_2, which determine directions $E_{1,2}$, are not perpendicular to one another. At the same time, eigenvectors of electrical induction $\mathbf{D}_{1,2}=\mathbf{E}_{1,2}+4\pi\hat{\varkappa}\mathbf{E}_{1,2}$ appear mutually perpendicular.

The direction of energy flow of the natural wave is characterized by the vector [EH] or the beam vector collinear with it s:

$$\mathbf{s}=\frac{\left(\frac{\partial\omega}{\partial\mathbf{k}}\right)}{\left(\frac{\partial\omega}{\partial\mathbf{k}}\right)^2}, \tag{2.9}$$

the modulus of which is equal to the value opposite to the group speed. Eigenvectors D, E, k and s are located in one plane perpendicular to H. Their locations for one of the natural waves are shown in Fig. 2-1. The beam vector obeys the relationship which can be obtained by multiplying scalarly (2.8) by e and differentiating the obtained equality with respect to k. We have:

$$2\omega\,\mathbf{e}^2+8\pi\omega\,\mathbf{e}\hat{\varkappa}\mathbf{e}+4\pi\omega^2\mathbf{e}\frac{\partial\hat{\varkappa}}{\partial\omega}\mathbf{e}=2c^2\mathbf{s}\left[\mathbf{e}\,[\mathbf{ke}]\right]. \tag{2.10}$$

In the derivation of (2.10) there was used the relationship:

$$\frac{\partial}{\partial\mathbf{k}}\mathbf{e}\left[\mathbf{k}\,[\mathbf{ke}]\right]=-2\left[\mathbf{e}\,[\mathbf{ke}]\right]. \tag{2.10a}$$

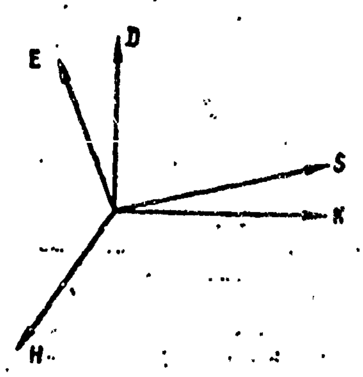

Fig. 2-1. Directions of vectors in an anisotropic medium.

2.2 First Approximation ($\mu \neq 0$); Truncated Equation for the Absorbing Medium

Let us assume now that the tensor of the linear medium contains not only the real but also imaginary part so that

$$\hat{\varkappa}(\omega) = \hat{\varkappa}^{R}(\omega) - i\mu \frac{\hat{\sigma}(\omega)}{\omega}, \tag{2.11}$$

We will consider that the medium is excited by a wave of the form (2.7) harmonic in time, and the polarization of the wave excited in the medium is similar to one of the natural polarizations of the nondissipative medium; it is required to determined the law of the change in the overall amplitude of the wave in space. Although the problem at hand for the linear medium examined at this point can be solved accurately (and for waves of a more complex form the solution can be written with the help of the Fourier integral), we will discuss the method of its approximate solution, which leads to replacement of the accurate equation (2.2) by an approximate first order equation for a slowly changing amplitude. As we will be convinced subsequently, the advantage of such an approach is, first of all, the possibility of its generalization of the nonlinear medium; at the same time this approach proves to be efficient in the solution of linear problems connected with the propagation of modulated waves (see, for example, Chapter V).

Thus, the presence in equation (2.2) of disturbances $\sim\mu$ leads to distinctions in the solution of the perturbed equation from the solution corresponding to $\mu = 0$ and having the form of a wave of constant amplitude (2.7) for which $\frac{dA_0}{dr} = 0$. With this the solution of the perturbed equation for $\mu = 0$ should, obviously, turn into a solution of the type (2.7). Having all of this in mind we will look for the general solution of the perturbed equation in the form of decomposition (see also [53]):

$$\mathbf{E} = \{\mathbf{e}_1 A(\mu r) e^{-ikr} + \mu \mathbf{U}_1(\mathbf{r}) + \mu^2 \mathbf{U}_2(\mathbf{r}) + \mu^3 \mathbf{U}_3(\mathbf{r}) + \cdots\} e^{i\omega t}, \tag{2.12}$$

where U_1, $U_2 \ldots$ are periodic functions of r, and quantity A, in contrast to the case $\mu = 0$. is no longer constant, and is determined by the differential equation

$$\frac{dA}{dr} = \mu B_1(A) + \mu^2 B_2(A) + \cdots \tag{2.13}$$

Now the problem is reduced to the determination of functions U_1, U_2, ..., B_1, B_2, ..., such that expression (2.12), after substitution into it of values of A, determined from (2.13), proved to be the solution of the initial differential equation (2.2).

The general procedure of finding the indicated functions is discussed in monograph [53] quoted above; one should note, however, that in practice due to the rapid growth in calculating difficulties with an increase in the number of terms of decompositions (2.12)-(2.13), it is necessary to be limited to the finding of only one to two first terms. Therefore, in being limited to m terms in decompositions ("m-approximation"), it is possible to state the problem of detecting of the approximate solution, i.e., such functions U_1 .. , U_m, B_1, ..., B_m, which would allow to obtaining a solution satisfying the initial equation (2.2) to within magnitudes of the order of μ^{m+1}. Here deviation of the thus obtained approximate solution from the exact one has an order of $\mu^{m+1}r$, and, consequently, can be made very small even at quite large r, if μ is small.

Not discussing here the special mathematical questions connected with asymptotic properties of the construct solutions (for more detail on this see monograph [53] and the mathematical work on the theory of differential equations containing a small parameter), we will pursue the analysis of the first approximation in the solution of (2.12), i.e., look for the solution of (2.2) correct to terms $\sim\mu^2$:

$$\mathbf{E} = \left[\mathbf{e}A(\mu\mathbf{r})\, e^{-ikr} + \mu\, \mathbf{U}(\mathbf{r})\right] e^{i\omega t}; \tag{2.14}$$

$$\frac{dA}{dr} = \mu\, \mathbf{B}(A). \tag{2.15}$$

Substituting (2.14) and (2.11) into (2.2), we have for separate components of the equation correct to terms of the second order with respect to μ:

$$\mathbf{P} = \left\{\hat{\varkappa}(\omega)\,\mathbf{e}\, A(\mu\mathbf{r})e^{-ikr} + \mu\hat{\varkappa}^{\kappa}(\omega)\,\mathbf{U}(\mathbf{r})\right\} e^{i\omega t}; \tag{2.16}$$

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = -\omega^2\mathbf{E}; \quad \frac{\partial^2\mathbf{P}}{\partial t^2} = -\omega^2\mathbf{P}; \tag{2.17}$$

$$[\nabla[\nabla \mathbf{E}]] = -\{[\mathbf{k}[\mathbf{ke}]]\, A e^{-i\mathbf{kr}} + i\mu[\mathbf{k}[\nabla \mathbf{e} A]]\, e^{-i\mathbf{kr}} + i\mu[\nabla[\mathbf{ke}A]]\, e^{-i\mathbf{kr}} - \mu[\nabla[\nabla \dot{\mathbf{U}}]]\}\, e^{i\omega t}. \tag{2.18}$$

Using (2.16-(2.18) and considering that quantity [k[ke]] can be determined from (2.8) (e is close in conditions to the eigenvector), we obtain:

$$c^2[\nabla[\nabla \mathbf{U}]] - \omega^2\{1 + 4\pi\hat{\varkappa}^s(\omega)\}\,\dot{\mathbf{U}} = i\{c^2[\mathbf{k}[\nabla \mathbf{e} A]] + c^2[\nabla[\mathbf{ke}A]] - 4\pi\omega\hat{\sigma}\mathbf{e}A\}\, e^{-i\mathbf{kr}}. \tag{2.19}$$

From the last relationship function B(A) and, consequently, the form of the first order equation, which determine the complex amplitude A can be simply determined. Actually, the linear differential operator, which acts on vector U in the left side of (2.19), has the eigenvalue $-i$k, and therefore the right side of (2.19) is a resonance force for it. At the same time, all functions of U_n in (2.12), in virtue of their determination in the method of successive approximations, should be limited for arbitrary r. For this it is necessary that the scalar product of the right side of (2.19) on e be equal to zero. (Polarization of vector U is perpendicular to e).[1] After multiplication by e we obtain the ordinary differential first order equation

$$\{\mathbf{e}[\mathbf{ke}]\}\, \nabla A + \mathbf{e}\hat{\alpha}\mathbf{e}A = 0, \tag{2.20}$$

where tensor $\hat{\alpha} = \frac{2\pi\omega\hat{\sigma}}{c^2}$.

In the derivation of (2.20) there is used the relation

$$\mathbf{e}[\mathbf{k}[\nabla \mathbf{e}]] + \mathbf{e}[\nabla[\mathbf{ke}]] = -2[\mathbf{e}[\mathbf{ke}]]\,\nabla. \tag{2.21}$$

Equation (2.20) is the sought, so-called "shortened" equation, which describes in the first approximation the change in complex amplitude A in space. Let us note that in the first approximation it is possible, in general, to be limited to the consideration of only the truncated equation (2.20), inasmuch as calculation of the term

[1]This requirement is analogous to the requirement usually used in the nonlinear theory of oscillations of systems with concentrated constants, i.e., the requirement of the absence of the first time harmonics in corresponding scalar functions $U_1, \ldots, U_n$

μU in (2.14) essentially does not change the results. Actually, relations (2.14)-(2.15) are written correct to terms $\sim\mu^2$; here the complex amplitude A, obtained from the thus truncated relations, on length r can be deflected from the exact value by quantity $\sim\mu^2 r$. On the other hand, as was noted above, amplitude A can be substantially changed only on intervals $\tilde{z}_M \sim \frac{\lambda}{\mu}$. Consequently, on the interval $\tilde{z}_M$ errors in the determination of complex amplitudes prove to be of the order of $\sim\mu$. Therefore, if we are interested, in the first place in the flow of transient processes in a nonlinear medium, it is possible not to consider the addend in (2.14) having an order of μ.

Let us turn now to an analysis of equation (2.20). Let us copy it in the form:

$$[\mathbf{e}[\mathbf{ke}]]\nabla A = -\mathbf{e}\hat{\alpha}\mathbf{e}A. \quad (2.22)$$

First of all, let us note that the vector $[\mathbf{e}[\mathbf{ke}]]$ has direction of the vector of the energy flow and, consequently, beam vector s, and its modulus is equal to

$$|[\mathbf{e}[\mathbf{ke}]]| = k\cos\widehat{\mathbf{ks}}, \quad (2.23)$$

where $\widehat{\mathbf{ks}}$ designates the angle between vectors k and s. In order to determine the law of the change in A in space, let us select a certain direction the unit vector along which will be designated by l_0, and the corresponding coordinate through l. Then the differential operator $G(A)$, standing in the left side of (2.22), can be convert to the form:

$$G(A) = k\cos\widehat{\mathbf{ks}}\cdot\cos\widehat{\mathbf{sl}_0}\cdot\frac{dA}{dl}. \quad (2.24)$$

Hence it is clear that the main direction of action of the differential operator G is the direction of the beam vector s. The solution of equation (2.22) has the form:

$$A = F([\mathbf{sr}])\exp[-\delta(\mathbf{sr})], \quad (2.25)$$

where $\delta = \frac{\mathbf{e}\hat{\alpha}\mathbf{e}}{|\mathbf{s}|k\cos\widehat{\mathbf{ks}}}$, and F is the arbitrary function of argument [sr], on which, in accordance with (2.24), the operator G does not act

In the boundary value problem, the direction of the change in amplitude is assigned directly conditions of the problem. Actually, if there is examined the semilimited anisotropic medium, onto which from without falls the monochromatic wave, the direction of the change in amplitude coincides, obviously, with the normal to the boundary. Introducing the cartesian coordinates and directing the axis z along the normal (here there can be any position of the optical axes with respect to the z axis) the solution of equation (2.21) can be represented in the form:

$$A = \exp\left\{-\delta \frac{z}{\cos \widehat{sz}}\right\} f(x, y), \tag{2.26}$$

where $f(x, y)$ – certain function determined by the boundary conditions. Formulas (2.25)-(2.26) give the solution of the problem at hand.

In the problem examined above about the propagation of unmodulated waves is a dissipative medium, the truncated equation (2.21), equivalent to within μ^2 to the initial second order equation in partial derivatives, proves to be an ordinary first order equation, which is absolutely similar to that which takes place in the theory of systems with concentrated constants (in this meaning we usually indicate the space-time analogy, see § 6 of this chapter). This case, of course, is the simplest; in general the presence of two independent variables in wave problems makes them more diverse than corresponding problems in the theory of systems with concentrated constants. The procedure stated above of obtaining shortened equations can easily be generalized in the case of modulated waves. Being limited by frames of the method of slowly changing amplitudes, we will consider that the changing of complex amplitudes with time are slow and, consequently, the solution of equation (2.2) can be sought in the first approximation in the form:

$$\mathbf{E} = \mathbf{e}A(\mu t, \mu \mathbf{r}) e^{i(\omega t - \mathbf{kr})} + \mu \mathbf{U}(\mathbf{r}, t), \tag{2.27}$$

where U is the periodic function of time and coordinate.

Repeating the procedure used in the derivation of (2.21) (here, however, in contrast to the case of the unmodulated wave the relation of the type (2.19) will be multiplied scalarly by $\tilde{\mathbf{e}} \exp i(\omega t - \mathbf{kr})$,

we arrive at the partial differential truncated equation of first order:

$$\left\{\left[2\omega\mathbf{e}+8\pi\omega\hat{\varkappa}^{\pi}\mathbf{e}+4\pi\omega^2\frac{\partial\hat{\varkappa}^{\pi}}{\partial\omega}\mathbf{e}\right]\frac{\partial A}{\partial t}-c^2\left[\mathbf{k}\left[\nabla\mathbf{e}A\right]\right]-\right.$$
$$\left.-c^2\left[\nabla\left[\mathbf{k}\mathbf{e}A\right]\right]+4\pi\omega\hat{\sigma}\mathbf{e}A\right\}\mathbf{e}=0. \tag{2.28}$$

The derivative $\frac{\partial\varkappa^{\pi}}{\partial\omega}$ in (2.28) appears from expression for P, which for the modulated wave can be represented in the first approximation as:

$$\mathbf{P}=\hat{\varkappa}(\omega)\,\mathbf{e}\,Ae^{i(\omega t-\mathbf{kr})}-\mu\mathbf{e}\frac{\partial A}{\partial t}e^{i(\omega t-\mathbf{kr})}\int_0^\infty t'\hat{\varkappa}^{\pi}(t')\,e^{-i\omega t'}dt'+$$
$$+\mu\int_0^\infty\hat{\varkappa}^{\pi}(t')\,\mathbf{U}(\mathbf{r},t-t')\,dt'=\hat{\varkappa}(\omega)\,\mathbf{e}\,Ae^{i(\omega t-\mathbf{kr})}-$$
$$-i\mu\frac{\partial\hat{\varkappa}^{\pi}}{\partial\omega}\cdot\mathbf{e}\cdot\frac{\partial A}{\partial t}\cdot e^{i(\omega t-\mathbf{kr})}+\mu\int_0^\infty\hat{\varkappa}^{\pi}(t')\,\mathbf{U}(\mathbf{r},t-t')\,dt'. \tag{2.29}$$

Replacing in (2.28) the coefficient with $\frac{\partial A}{\partial t}$ with the help of (2.10), we have finally:

$$\left[\mathbf{e}\left[\mathbf{k}\mathbf{e}\right]\right]\mathbf{s}\frac{\partial A}{\partial t}+\left[\mathbf{e}\left[\mathbf{k}\mathbf{e}\right]\right]\nabla A+\mathbf{e}\hat{\alpha}\mathbf{e}A=0. \tag{2.30}$$

The general solution of (2.30) is the product of the solution of the stationary equation (2.21) on a certain function of the argument $\mu(t - \mathbf{sr})$:

$$A=A(\mu\mathbf{r})\cdot f\left[\mu\cdot(t-\mathbf{sr})\right]. \tag{2.31}$$

Further we will pu...ue the generalization of truncated equations of the form (2.21) or (2.30) in the case of a nonlinear medium. Equation (2.30) proves to be very convenient in the investigation of modulated waves in linear dispersive media, in particular, in the investigation of distortions of modulation in a dispersive medium, where calculations founded on spectral concepts prove to be more laborious.

§ 3. Interaction of Waves in a Nonlinear Anisotropic Media

3.1. Quadratic Medium. Truncated Equations.

As was already indicated in Chapter I, the appearance of a wave of nonlinear polarization in a quadratic medium is the result

of the interaction of two waves of the field; in this paragraph, by the method of slowly changing amplitudes, we investigate the general regularities of three-frequency interaction in space. Here for the fullest description of such an interaction, one should consider not only the two initial waves of the field but also the natural wave of the medium on the combination frequency. In accordance with what has been said let us present field in the nonlinear medium in the form:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \mathbf{e}_1 A_1 (\mu t, \mu \mathbf{r}) e^{i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} + \mathbf{e}_2 A_2 (\mu t, \mu \mathbf{r}) e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} + \mathbf{e}_3 A_3 (\mu t, \mu \mathbf{r}) e^{i(\omega_3 t - \mathbf{k}_3 \mathbf{r})} + \text{ complex conjugate.} \quad (2.32)$$

(Here we will no longer repeat in detail the procedure of the derivation of the truncated equations, and therefore we will not write vectors $U_n \sim \mu^n$ in (2.32).) In (2.32) vectors e_n characterize the polarizations of the waves, A_n – complex amplitudes, and k_n – wave vectors of natural waves of the medium. Between frequencies ω_n for the examined interaction there takes place the relation:

$$\omega_1 + \omega_2 = \omega_3. \quad (2.33)$$

Waves (2.32) in ahe quadratic medium excite pairwise forced waves of nonlinear polarization on combination frequencies. Amplitudes of these forced waves have the form:

$$P^{\omega_1+\omega_2} \sim A_1 A_2; \; P^{\omega_3-\omega_2} \sim A_3 A_2^*; \; P^{\omega_3-\omega_1} \sim A_3 A_1^* \text{ etc.}$$

Waves of nonlinear polarization excite corresponding waves of the field, and the latter, in turn, new waves of polarization; all of this determines the interaction of waves E_1, E_2 and E_3 in the quadratic medium. Let us recall (see the introduction, formula (I.23) and (I.24)) that the thus appearing nonlinear interactions can lead to stored effects only in the case when wave numbers of forced waves of nonlinear polarization are close to wave numbers of natural waves of the medium on corresponding combination frequencies. The condition of the appearance of stored effects with

three-frequency interactions (it is accepted to call it the "condition of synchronism") has, obviously, the form

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3. \tag{2.34}$$

Subsequently, we will consider also small $/\sim\mu/$ deviations from the exact condition of synchronism; we will consider that between wave vectors k_n there takes place a relationship somewhat more general than (2.34) of the form:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \Delta, \tag{2.35}$$

where $|\Delta|/k \sim \mu$.

Let us turn to the derivation of the truncated equations. For this, just as in § 2, the unknown solution of (2.32) should be substituted into equation (2.2). Let us note that here, in contrast to the linear medium, the vector of polarization P will contain not only terms of the form (2.29), taken for frequencies ω_1, ω_2 and ω_3 (we will designate them {1}, {2} and {3}), but also nonlinear terms determined by the interaction of the waves. Therefore, the full expression for vector P in the quadratic medium, excited by the three waves, has the following form (in P only components having frequencies ω_1, ω_2 and ω_3 are increased in value):

$$\begin{aligned}\mathbf{P} = \{1\} + \{2\} + \{3\} + \hat{\chi}^{\omega_1+\omega_2}\mathbf{e}_1\mathbf{e}_2 A_1 A_2 e^{i(\omega_3 t - \mathbf{k}_3\mathbf{r} - \Delta\mathbf{r})} + \\ + \hat{\chi}^{\omega_3-\omega_1}\mathbf{e}_3\mathbf{e}_1 A_3 A_1^* e^{i(\omega_2 t - \mathbf{k}_2\mathbf{r} + \Delta\mathbf{r})} + \hat{\chi}^{\omega_3-\omega_2}\mathbf{e}_3\mathbf{e}_2 A_3 A_2^* e^{i(\omega_1 t - \mathbf{k}_1\mathbf{r} + \Delta\mathbf{r})} + \\ + \text{complex conjugate.}\end{aligned} \tag{2.36}$$

Conducting further computations, just as in the preceding paragraph, and collecting terms corresponding to identical frequencies, (for this one should conduct term-by-term integration with respect to periods $T_{1,2,3} = \frac{2\pi}{\omega_{1,2,3}}$) , we arrive at truncated equations of the form

$$\begin{aligned}[\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\, s_1 \frac{\partial A_1}{\partial t} + [\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\nabla A_1 + (\mathbf{e}_1\hat{\alpha}_1\mathbf{e}_1)A_1 + \\ + i\beta\omega_1^2 e^{+i\Delta\mathbf{r}} A_3 A_2^* = 0; \\ [\mathbf{e}_2[\mathbf{k}_2\mathbf{e}_2]]\, s_2 \frac{\partial A_2}{\partial t} + [\mathbf{e}_2[\mathbf{k}_2\mathbf{e}_2]]\nabla A_2 + (\mathbf{e}_2\hat{\alpha}_2\mathbf{e}_2)A_2 + \\ + i\beta\omega_2^2 e^{+i\Delta\mathbf{r}} A_3 A_1^* = 0;\end{aligned} \tag{2.37}$$

$$[\mathbf{e}_3[\mathbf{k}_3\mathbf{e}_3]]\,\mathbf{s}_3\frac{\partial A_3}{\partial t}+[\mathbf{e}_3[\mathbf{k}_3\mathbf{e}_3]]\,\nabla A_3+(\mathbf{e}_3\hat{\alpha}_3\mathbf{e}_3)A_3+$$
$$+i\beta\omega_3^2e^{-i\Delta r}A_1A_2=0,$$

where

$$\beta=\frac{2\pi}{c^2}(\mathbf{e}_1\hat{\chi}^{\omega_3,-\omega_2}\mathbf{e}_3\mathbf{e}_2)=\frac{2\pi}{c^2}(\mathbf{e}_2\hat{\chi}^{\omega_3,-\omega_1}\mathbf{e}_3\mathbf{e}_1)=\frac{2\pi}{c^2}(\mathbf{e}_3\hat{\chi}^{\omega_1,+\omega_2}\mathbf{e}_1\mathbf{e}_2). \qquad (2.38)$$

The last equalities take place in virtue of the relationship (1.56). With an accuracy of $\sim\mu^2$ equations (2.37) are equivalent to the initial equation (2.2) for the case of the three-frequency interaction. When $\beta = 0$ equations (2.37) become independent; each of them has the form of equation (2.30) - in this case the medium is linear, and the principle of superposition acts. Conversely, when $\beta \neq 0$ waves of different frequencies interact with each other; the process of interaction is described the last in (2.37) having an order of $\sim\mu$. As one should have been led to expect, the value of these terms is determined not only by the nonlinearity of the medium β and amplitudes of interacting waves but also by the dispersion properties of the medium, which enter into nonlinear terms through exponentials of the form $\exp(\pm i\Delta r)$. Here the maximum nonlinear interaction takes place, obviously, when $\Delta r \equiv 0$ (this corresponds, in particular, to the fulfillment of the exact condition of synchronism (2.34)). At large $|\Delta|$ nonlinear terms prove to be rapidly oscillatory and therefore cannot essentially change the complex amplitudes A_n; waves propagate practically just as they do in a linear medium. Although equation (2.37) is simpler than the initial nonlinear equation, even here an analytic solution in general is not possible to obtain. We will pursue the analysis further and, where it is possible, by secondary simplifications of system (2.37); here we will deduce certain general relationships taking place for the three-frequency interaction of unmodulated waves in a quadratic medium without losses.

Let us assume that the nonlinear dielectric occupies the half-space $z > 0$, and three plane waves of the type (2.32) drop on it at different angles from the vacuum. Then the complex amplitudes of the waves, which passed into the dielectric, depend

obviously only on z, and equations (2.37) obtain the form:

$$k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_1}{dz} + i\beta\omega_1^2 e^{+i\Delta_z z} \cdot C \cdot A_3 A_2^* = 0; \quad (2.39a)$$

$$k_2 \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \widehat{\mathbf{s}_2 \mathbf{z}_0} \frac{dA_2}{dz} + i\beta\omega_2^2 e^{+i\Delta_z z} \cdot C \cdot A_3 A_1^* = 0; \quad (2.39b)$$

$$k_3 \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cdot \cos \widehat{\mathbf{s}_3 \mathbf{z}_0} \frac{dA_3}{dz} + i\beta\omega_3^2 e^{-i\Delta_z z} \cdot C \cdot A_1 A_2 = 0, \quad (2.39c)$$

where z_0 – unit vector in the direction of the z axis,

$$C = \exp\{i(\Delta_x x + \Delta_y y)\}. \quad (2.40)$$

From (2.39) and (2.40) there follows, thus, the remarkable conclusion: the effectiveness of the three-frequency interaction, carried out along the z axis, is affected only by z – component "vector of frequency difference" Δ and Δ_z.

Multiplying these equations by A_1^*/ω_1^2, A_2^*/ω_2^2 and A_3^*/ω_3^2 and adding them with complex conjugate expressions, after integration the following relationships can be obtained, which are correct for the arbitrary section z

$$\left.\begin{aligned} \frac{k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}}{\omega_1^2} A_1 A_1^* + \frac{k_3 \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cdot \cos \widehat{\mathbf{s}_3 \mathbf{z}_0}}{\omega_3^2} A_3 A_3^* = \text{const.} \\ \frac{k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}}{\omega_1^2} A_1 A_1^* + \frac{k_2 \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \widehat{\mathbf{s}_2 \mathbf{z}_0}}{\omega_2^2} A_2 A_2^* = \text{const.} \end{aligned}\right\} \quad (2.41)$$

In order to present (2.41) in a more transparent form, we will consider that the amplitude of the magnetic field strength H is expressed by amplitude A in the following way:

$$H = \frac{c}{\omega} k A \cos \widehat{\mathbf{ks}}. \quad (2.42)$$

Then relations (2.41) can be presented in the form

$$\frac{[\mathbf{E}_1 \mathbf{H}_1^*]\, \mathbf{z}_0}{\omega_1} + \frac{[\mathbf{E}_3 \mathbf{H}_3^*]\, \mathbf{z}_0}{\omega_3} = \text{const}; \quad \frac{[\mathbf{E}_1 \mathbf{H}_1^*]\, \mathbf{z}_0}{\omega_1} - \frac{[\mathbf{E}_2 \mathbf{H}_2^*]\, \mathbf{z}_0}{\omega_2} = \text{const}. \quad (2.43)$$

Subtracting the second relations of (2.43) from the first, we have:

$$\frac{[E_2 H_2^*]\, z_0}{\omega_2} + \frac{[E_3 H_3^*]\, z_0}{\omega_3} \doteq \text{const.} \tag{2.43a}$$

From (2.43) there follows the law of conservation of energy flow. Let us multiply the first relation (2.43) by ω_1, relation (2.43a) by ω_2 and add the obtained expressions. Considering (2.33) we will obtain:

$$[E_1 H_1^*]\, z_0 + [E_2 H_2^*]\, z_0 + [E_3 H_3^*]\, z_0 = \text{const.} \tag{2.44}$$

The last one means that the general energy flow through the area element parallel to the boundary does not depend on the coordinate z.

3.2. Energy Relationships with Three-Frequency Interactions in a Quadratic Medium. Discussion.

The general energy relations (2.41)-(2.44), which characterize the flow of three-frequency interactions in a quadratic medium, allow a very graphic quantum interpretation. Actually, in quantum language, the excitation of harmonics and combination frequencies should be treated, obviously, as processes of the merging and division of photons. Having this in mind, the relations (2.33) and (2.34) multiplied by Planck's constant should be interpreted as laws of the conservation of energy and momentum in an elementary three-photon interaction. Relations (2.43) mean that the sum of the number of quanta of frequencies ω and ω_3 and the difference in the number of quanta of frequencies ω_1 and ω_2, which passed through a unit area element parallel to the border of the dielectric (in an anisotropic dielectric quanta move along the beam vector), remain constant. Let us note that in relations (2.33), (2.34) and (2.43) nonlinear properties of the medium (in quantum language, they determine the probability of the merging or division of photons), in general, do not appear; therefore, these relations act in all cases when three-photon interactions are solved. Quasi particles corresponding to the interacting fields should not have to be

photons; they can also be phonons, magnons etc. The fulfillment of relations (2.33), (2.34) and (2.43) will be, of course, obligatory for them (see also [103]). In connection with what has been said, there is interest in the consideration of nonlinear interactions of electromagnetic waves by methods quantum electrodynamics; certain results in this direction are contained in [98 and 101].

Finally, relations (2.43) prove to be analogous in form (for more detail on this see § 6 of this chapter) to the well-known concentrated constants in the theory of nonlinear reactive systems, the so-called Manley-Rowe relationships [104]. From this point of view it is possible to examine (2.43) as a generalization of Manley-Rowe relationships on anisotropic media (for a one-dimensional medium such a generalization was carried out in [106 and 107]) and the quantum interpretation (2.43) given above as the quantum treatment of the relationships of Manley-Rowe.[1]

Subsequently, in specific problems, we will use widely relationships of Manley-Rowe; here the values of constants in their right sides can be determined with the help of boundary conditions characteristic for the given problem.

3.3. On the Interaction of Waves in a Cubic Medium

We will now examine the process of the interaction of waves in an anisotropic dispersive medium, the lowest term in the decomposition of field polarization for which is the cubic term. The method of derivation of truncated equations here does not differ from that examined in 3.1; therefore, being interested, in the first place, only in qualitative effects distinguishing the cubic medium from the quadratic, here we will not examine the general case of the four-frequency interaction in the cubic medium but will limit ourselves to an analysis of the degenerated interaction of unmodulated waves, which allow revealing the most characteristic properties of the cubic medium.

[1]For systems with concentrated constants such quantum interpretation was first given by Weiss [105].

Let us examine the interaction in the cubic medium of two waves with frequencies

$$\omega_1 = \omega \text{ and } \omega_2 = 3\omega, \tag{2.45}$$

$$\mathbf{E} = \mathbf{e}_1 A_1(\mu \mathbf{r}) e^{i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} + \mathbf{e}_2 A_2(\mu \mathbf{r}) e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} + \text{complex conjugate} \tag{2.46}$$

The relationship between wave vectors of natural waves of the linear medium k_1 and k_2 for the examined interaction should be recorded in the form

$$\mathbf{k}_2 = 3\mathbf{k}_1 + \Delta, \tag{2.47}$$

where $|\Delta|/k \sim \mu$.

In § 3 of Chapter I it was shown that with the passage of two waves of frequencies ω_1 and ω_2 through a cubic medium, in it there appear components of nonlinear polarization at frequencies $(\omega_1+\omega_1+\omega_1) = 3\omega_1$; $(\omega_1+\omega_1-\omega_1) = \omega_1$; $(\omega_1+\omega_2-\omega_1) = \omega_2$; $(\omega_2-\omega_1-\omega_1) = \omega_2 - 2\omega_1$; $(\omega_1+\omega_2-\omega_2) = \omega_1$; and $(\omega_2+\omega_2-\omega_2) = \omega_2$. Under the condition (2.45) enumerated components have a frequency ω or 3ω. The full expression for the vector of polarization P (only components with frequencies ω and 3ω are retained) now has the form (compare with formula (2.36)

$$\begin{aligned}\mathbf{P} = (1) + (2) &+ \hat{\theta}^{\omega_1+\omega_1+\omega_1} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_1 A_1^3 e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} e^{i\Delta \mathbf{r}} + \\ &+ \hat{\theta}^{\omega_1+\omega_1-\omega_1} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_1 A_1^2 A_1^* e^{i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} + \\ &+ 2\hat{\theta}^{\omega_1+\omega_2-\omega_1} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 A_1 A_1^* A_2 e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} + \\ &+ \hat{\theta}^{\omega_1+\omega_1-\omega_2} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 A_1^2 A_2^* e^{-i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} e^{-i\Delta \mathbf{r}} + \\ &+ 2\hat{\theta}^{\omega_1+\omega_2-\omega_2} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 A_1 A_2 A_2^* e^{i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} + \\ &+ \hat{\theta}^{\omega_2+\omega_2-\omega_2} \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_2 A_2^2 A_2^* e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} + \text{complex conjugate.}\end{aligned} \tag{2.48}$$

Proceeding further in the same way as in the preceding paragraphs, it is possible to arrive at truncated equations, which describe the process of interaction of unmodulated waves in a nondissipative cubic medium. They have the form

$$\begin{aligned}&k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_1}{dz} + i3C\gamma\omega_1^2 A_1^{*2} A_2 e^{i\Delta_z z} + \\ &\quad + i\gamma_1 \omega_1^2 A_1^2 A_1^* + i\gamma_2 \omega_1^2 A_1 A_2 A_2^* = 0; \\ &k_2 \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \widehat{\mathbf{s}_2 \mathbf{z}_0} \frac{dA_2}{dz} + i\gamma\omega_2^2 A_1^3 e^{i\Delta_z z} \cdot C + \\ &\quad + i\gamma_2 \omega_2^2 A_1 A_1^* A_2 + i\gamma_3 \omega_2^2 A_2^2 A_2^* = 0.\end{aligned} \tag{2.49}$$

Here

$$C = e^{i(\Delta_x x + \Delta_y y)}; \quad \gamma_1 = \frac{2\pi}{c^2} (\mathbf{e}_1 \hat{\theta}^{\omega_1+\omega_1-\omega_1} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_1);$$

$$\gamma_2 = \frac{2\pi}{c^2} (\mathbf{e}_2 \hat{\theta}^{\omega_2+\omega_2-\omega_2} \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_2);$$

$$\gamma = \frac{2\pi}{c^2} (\mathbf{e}_2 \hat{\theta}^{\omega_1+\omega_1+\omega_1} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_1) = \frac{2\pi}{3c^2} (\mathbf{e}_1 \hat{\theta}^{\omega_2-\omega_1-\omega_1} \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_1);$$

$$\gamma_3 = \frac{4\pi}{c^2} (\mathbf{e}_1 \hat{\theta}^{\omega_1+\omega_2-\omega_2} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2) = \frac{4\pi}{c^2} (\mathbf{e}_2 \hat{\theta}^{\omega_1-\omega_1+\omega_2} \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2). \tag{2.50}$$

The last equalities follow from relations (1.37).

The most important distinction of truncated equations (2.49) from corresponding equations of the quadratic medium (2.39) is the fact that each of equations (2.49) contains no longer one by one, as in (2.39), but three nonlinear terms describing the interaction of the waves. Here the character of nonlinear interactions described by various nonlinear terms in (2.49), as it is easy to see, is different. Really, nonlinear interactions proportional to the coefficient γ, just as nonlinear interactions in a quadratic medium, considerably depend on phase relationships between the waves, which is described by the factor $e^{i\Delta_z z}$. Nonlinear interactions proportional to γ_1, γ_2, and γ_3, are not connected with the phase relationships and, consequently, also with dispersion properties of the medium.

Therefore, the interactions of the first type (just as analogous interactions in a quadratic medium, they are maximum when $\Delta_z = 0$ and practically unimportant when $\Delta_z \to \infty$) can be called "coherent" in contrast to "incoherent" interactions, which are connected with nonlinear coefficients γ_1, γ_2, and γ_3. Comparing (2.50) with (1.32) and (1.33), it is easy to clarify the physical meaning of "incoherent" interactions (and "self-actions") of electromagnetic waves in a cubic medium: they, obviously, are connected with nonlinear corrections to the dielectric constant.

We will give subsequently a detailed consideration of the pattern of "incoherent" interactions; here we will limit ourselves only to the derivation of general energy relations similar to relations (2.43)-(2.44).

Let us multiply the first equation (2.49) by $\frac{A_1^*}{3\omega_1^2}$, and the second by $\frac{A_2^*}{\omega_2^2}$, and let us add the obtained expressions with their complex conjugate. Integrating with respect to z, we obtain

$$\frac{k_1 \cos \hat{k_1 s_1} \cos \hat{s_1 z_0}}{3\omega_1^2} A_1 A_1^* + \frac{k_2 \cos \hat{k_2 s_2} \cos \hat{s_2 z_0}}{\omega_2^2} A_2 A_2^* = \text{const.} \tag{2.51}$$

Considering (2.42) and (2.45), we have

$$\frac{[\mathbf{E}_1 \mathbf{H}_1^*]\, z_0}{3\omega_1} + \frac{[\mathbf{E}_2 \mathbf{H}_2^*]\, z_0}{\omega_2} = \text{const} \tag{2.52}$$

– Manley-Rowe relationship, which is fulfilled in every section z of the cubic medium for the degenerated four-frequency interaction.

From (2.52) it follows that the increase in quantity of photons of frequency ω_2 passing through a unit area element, parallel to the border of the dielectric, by a certain number ΔN_2 is inevitably connected with a decrease in the number of photons of frequency ω_1 by $\Delta N_1 = 3\Delta N_2$ and conversely. Just as in the case of the quadratic medium, the indicated relation would have been possible to write using the quantum interpretation of nonlinear interactions of waves as a basis.

Using (2.52) and (2.45), we will obtain the law of the conservation of energy flow:

$$[\mathbf{E}_1 \mathbf{H}_1^*]\, z_0 + [\mathbf{E}_2 \mathbf{H}_2^*]\, z_0 = \text{const.} \tag{2.53}$$

§ 4. General Characteristic of Interactions of Waves in Nonlinear Dispersive Media. Boundary Value Problems. Secondary Simplifications of Truncated Equations. Side Forces in a Nonlinear Medium

4.1. Boundary Value Problems; Classification of Nonlinear Interactions

Truncated equation (2.37) or (2.39) and (2.49) describe the interactions of waves in a nonlinear medium, which occur in absence

of side fields (the linear problem is uniform).[1]

In § 3, in the derivation of truncated equations, boundary conditions were considered by us only in the determination of direction, along which there occurs a change in complex amplitudes, and the actual values of complex amplitudes A_n on the border of the nonlinear medium were not specified. A specific definition of the conditions permits separating within the bounds of the three-frequency (for a square medium) or four-frequency (for a cubic medium) interactions different special cases corresponding to different physical effects. A detailed investigation of the boundary value problems are given in Chapters III-V; here we will give only their classification and also examine certain general regularities of the course of nonlinear interactions corresponding to various boundary conditions.

In the analysis of general properties of nonlinear interactions, energy relations of the type (2.43)-(2.44) and (2.52)-(2.53) can be used very effectively; being interested only in the fundamental side of the matter, we will limit ourselves here to the consideration of interactions of unmodulated waves in the medium without losses, for the case $|\Delta| = 0$.

Let us turn to the three-frequency interactions described by system (2.39). In the absence of side fields, the three-frequency interaction can appear only in the case when on the border of the nonlinear medium, at least amplitudes of two waves are different from zero. Here, besides the general case,

$$A_1(0) \neq 0;\ A_2(0) \neq 0;\ A_3(0) \neq 0$$

one should examine such cases for which

$$A_1(0) \neq 0;\ A_2(0) \neq 0;\ A_3(0) = 0; \qquad (2.54)$$

$$A_1(0) = 0;\ A_2(0) \neq 0;\ A_3(0) \neq 0; \qquad (2.55)$$

$$A_1(0) \neq 0;\ A_2(0) = 0;\ A_3(0) \neq 0; \qquad (2.56)$$

[1]We call such a problem in nonlinear theory uniform, noting in this the absence of "linear" side forces.

From equations (2.39) it follows that with the three-frequency interaction in a quadratic medium, in general, there are two opposite processes:

1. Waves of frequencies ω_1 and ω_2 excite the wave at frequency ω_3 – the process of "merging of photons," described by equation (2.39c) occurs.

2. Simultaneously with the process of the merging of photons, there occurs the process of the "disintegration" of photons of frequency ω_3 (interaction of waves of frequencies ω_3 and ω_2 and frequencies ω_3 and ω_1) described by equations (2.39a) and (2.39b).

In accordance with (2.43)-(2.44), for increases in energies dW_n of waves on segment dz these relations take place: (in contrast to the integral relations (2.43)-(2.44), then can be called differential energy relations)

$$\frac{dW_1}{\omega_1} = \frac{dW_2}{\omega_2}; \quad \frac{dW_1}{\omega_1} = -\frac{dW_3}{\omega_3}; \quad \frac{dW_2}{\omega_2} = -\frac{dW_3}{\omega_3} \tag{2.57}$$

or, in quantum interpretation

$$dN_1 = dN_2; \; dN_1 = -dN_3; \; dN_2 = -dN_3, \tag{2.58}$$

where $N_1(z)$, $N_2(z)$ and $N_3(z)$ are numbers of photons in waves 1, 2, and 3. If $N_1(0) \approx N_2(0)$ and $N_3(0) = 0$ (in general, $N_3(0) \ll N_1(0), N_2(0)$) in any case for not too large z the process of merging dominates over the process of disintegration; a decrease in the number of photons in waves 1 and 2 in virtue of (2.57) is equal and unimportant and therefore here for a description of the nonlinear interaction only one equation (2.39c) is sufficient. This equation when $A_1 \simeq \text{const}, A_2 \simeq \text{const}$ has the form

$$k_3 \cos \mathbf{k}_3\hat{\ }\mathbf{s}_3 \cdot \cos \mathbf{s}_3\hat{\ }\mathbf{z}_0 \frac{dA_3}{dz} + i\beta\omega_3^2 C_1 = 0, \tag{2.59}$$

where constant $C_1 = A_1(0) \cdot A_2(0)$.

From (2.59) it follows that the monotonic growth of amplitude A_3 will take place as long as amplitudes A_1 and A_2 can be considered constants. Amplitude A_3 grows linearly in this case with coordinate

$$A_3(z) = A_3(0) + \frac{i\beta\omega_3^2 C_1 z}{k_3 \cos \widehat{k_3 s_3} \cos \widehat{s_3 z_0}}. \tag{2.59a}$$

At large z the reverse process (disintegration) becomes important. Here equations (2.39a, b) should be considered. The presence of the reverse process delays the rate of growth A_3 and, in general, can obviously lead even to a change in the sign of the derivative $\frac{d|A_3|}{dz}$ and, consequently, to the three-dimensional beats of the interacting waves.

If for boundary conditions of the type (2.54) at small z the process of "merging" of photons always dominates over the process of "disintegration," for conditions of the type (2.55)-(2.56) a reverse situation takes place. If $N_3(0) \gg N_{1,2}(0)$, for a description of the process of "disintegration" at small z, here instead of the full system (2.39) there can be used the equations

$$k_1 \cos \widehat{s_1 k_1} \cdot \cos \widehat{s_1 z_0} \frac{dA_1}{dz} + i\beta\omega_1^2 A_3(0) A_2^* = 0; \tag{2.60a}$$

$$k_2 \cos \widehat{k_2 s_2} \cdot \cos \widehat{s_2 z_0} \frac{dA_2}{dz} + i\beta\omega_2^2 A_3(0) A_1^* = 0. \tag{2.60b}$$

Differentiating equation (2.60a) with respect to z and substituting the derivative $\frac{dA_2^*}{dz}$ from the equation complex conjugate to equation (2.60b), we arrive at the second order equation for amplitude A_1:

$$\frac{d^2 A_1}{dz^2} - \frac{\beta^2 |A_3(0)|^2 \omega_1^2 \omega_2^2 A_1}{k_1 k_2 \cos \widehat{k_1 s_1} \cdot \cos \widehat{s_1 z_0} \cos \widehat{k_2 s_2} \cdot \cos \widehat{s_2 z_0}} = 0. \tag{2.61}$$

From (2.61) it follows that as long as the intensity of the wave on frequency ω_3 considerably exceeds the intensity of waves at frequencies ω_1 and ω_2, the "disintegration" of photons of frequency ω_3 leads to an exponential increase in amplitudes $A_{1,2}$ with the coordinate. The general solution of equation (2.61) has the form

$$A_1(z) = a_1 e^{\Gamma_1 z} + b_1 e^{-\Gamma_1 z}, \tag{2.62}$$

and factor of increase

$$\Gamma_1 = \sqrt{\frac{\beta^2 |A_3(0)|^2 \omega_1^2 \omega_2^2}{k_1 k_2 \cos \widehat{k_1 s_1} \cos \widehat{s_1 z_0} \cdot \cos \widehat{k_2 s_2} \cdot \cos \widehat{s_2 z_0}}}. \tag{2.63}$$

At sufficiently large $A_{1,2} \sim A_3$ it is necessary to consider the reverse process (in this case, this is the process of merging of the photons described by equation (2.39c); here the exponential growth in $A_{1,2}$ is delayed, and derivatives $\frac{d|A_{1,2}|}{dz}$ can even change sign.

Besides extreme cases indicated above, for which either $N_1(0), N_2(0) \gg N_3(0)$, or $N_3(0) \gg N_1(0), N_2(0)$, other relationships between the boundary amplitudes of interacting waves can be examined. In particular, with further specific definition of boundary conditions (2.54)-(2.56), there can be practical interest in cases when

$$N_2(0) \gg N_1(0),\ N_3(0) \tag{2.64}$$

and

$$N_2(0) \simeq N_3(0) \gg N_1(0). \tag{2.65}$$

For not too large z the process of the interaction of waves corresponding to boundary conditions (2.64) is described by these equations:

$$k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_1}{dz} + i\beta\omega_1^2 A_3 A_2^*(0) = 0; \tag{2.66}$$

$$k_3 \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cdot \cos \widehat{\mathbf{s}_3 \mathbf{z}_0} \frac{dA_3}{dz} + i\beta\omega_3^2 A_1 A_2(0) = 0. \tag{2.67}$$

Differentiating (2.66) with respect to z and substituting $\frac{dA_3}{dz}$ from (2.67) we arrive at the differential second order equations for A_1:

$$\frac{d^2 A_1}{dz^2} + \frac{-\beta^2 |A_2(0)|^2 \omega_1^2 \omega_3^2}{k_1 k_3 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \cdot \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cdot \cos \widehat{\mathbf{s}_3 \mathbf{z}_0}} \cdot A_1 = 0. \tag{2.68}$$

The general solution of equation (2.68) has the form

$$A_1 = a_2 e^{\Gamma_2 z} + b_2 e^{-\Gamma_2 z}, \tag{2.69}$$

where

$$\Gamma_2 = i \sqrt{\frac{\cdot\ \beta^2 |A_2(0)|^2 \omega_1^2 \omega_3^2}{k_1 k_3 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \cdot \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cdot \cos \widehat{\mathbf{s}_3 \mathbf{z}_0}}}. \tag{2.70}$$

From (2.69)-(2.70) it thus follows that in the case when amplitude A_2 can be considered constant, the process of the change in amplitudes A_1 and A_3 has a character of three-dimensional beats [to compare with (2.59a) and (2.62)].

Finally, initial stages of the process of three-frequency interaction with boundary conditions of the form (2.65) are described, obviously, by equation

$$k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_1}{dz} + i\beta\omega_1^2 A_3 A_2^* = 0. \tag{2.71}$$

In all examples examined till now, boundary conditions were selected in such a way that in any case near the border of the nonlinear medium, one of two possible nonlinear processes (disintegration or merging) played the dominating role, which inevitably caused a change in the space of amplitudes of the interacting waves. At the same time, in general, such a selection of boundary conditions is possible at which at each point of the nonlinear medium there occurs the dynamic equilibrium between processes of merging and disintegration and, consequently, amplitudes of interacting waves remain constant.

Relationships between amplitudes of such stationary waves can be found from equations (2.39), in which all derivatives $\frac{dA_k}{dz}=0\ (k=1, 2, 3)$. Presenting complex amplitudes in the form $A_k = |A_k| \exp i\varphi_k(z)$ and equating to zero separately the real and imaginary parts of the obtained relations (see also Chapter III), we arrive at the formula characterizing the bond between numbers of photons in stationary waves for an arbitrary point of the nonlinear medium (see equation (4.24)):

$$N_1(z) \cdot N_2(z) = [N_1(z) + N_2(z)] \cdot N_3(z). \tag{2.72}$$

The specific definition of boundary conditions permits in a number of cases considerably simplifying system of truncated equations. In the weakly nonlinear medium the spatial scales of processes of the change in amplitudes of interacting waves prove to be usually very large for processes described by equations (2.59), (2.61), (2.68) and (2.71). These scales have, obviously, (see formula (2.38)) the value

$$L_0^{(2)} \simeq \frac{1}{\chi A_n(0)\, k_n}. \tag{2.73}$$

Inasmuch as it was already indicated in the introduction (see formula (I.14)), quantity $\chi A(0) < 10^{-5} - 10^{-6}$ and $L_0^{(2)} \simeq 10^5 - 10^6 \lambda$, in many

problems of the characteristic length of the coherent interaction $l_{\text{к}}$

$$l_{\text{к}} < L_0^{(2)}. \qquad (2.74)$$

Therefore, although in principle both the process of merging and process of disintegration of the photons always take place, in the fulfillment of condition (2.74) it is possible, in general, to consider only that one of these processes which plays the determining role on the border of the nonlinear medium. Here the analysis of the full system of truncated equations can be replaced by an analysis of the system in which amplitudes of powerful waves are examined as assigned functions.

Subsequently, the approximation founded on the indicated circumstance will be called the approximation of the assigned field. Inasmuch as in the approximation of the assigned field interacting waves are disparate, here within the bounds of the three-frequency interaction it is possible to separate various physical effects corresponding to different boundary conditions on the border of the nonlinear medium.

In the fulfillment of condition (2.74) the problem with boundary conditions (2.54) can be called the problem on the radiation of harmonics and total frequencies in the quadratic medium. The problem with boundary conditions (2.55) or (2.56) when $N_3(0) \sim N_2(0)$ or $N_3(0) \sim N_1(0)$ is reduced to the detcting of radiation of waves of difference frequencies in the quadratic medium. Finally, nonlinear effects appearing in those cases when the intensity of one of the interacting waves considerably exceeds the intensities of the two others (see equations (2.60a)-(2.60b) and (2.66)-(2.67)), within the bounds of the approximation of the assigned field can be called the parameteric interaction of the waves. The last term is based on the fact that the analysis of interactions described by equations (2.60a)-(2.60b) or (2.66)-(2.67) can be conducted also on the basis of concepts about the medium, the dielectric constant of which is changed in space and in time according to the law determined by

changes in the field of the intense wave. In order to be convinced of this, we will derive equations describing the propagation of waves in the medium, the dielectric constant $\hat{\varepsilon}(t, \omega, z)$ of which is changed according to the law of the traveling wave:

$$\hat{\varepsilon}(t, \omega, z) = \hat{\varepsilon}_0(\omega) + \mu \hat{\varepsilon}_1 \left\{ e^{i(\omega_3 t - k_3 z)} + e^{-i(\omega_3 t - k_3 z)} \right\}. \qquad (2.75)$$

Here u, just as before, small parameter $(\mu \ll 1)$, the wave vector $k_3 = k_3 z_0$. We will consider that in the examined medium at arbitrary angles to the normal, directed along the z axis, waves at frequencies ω_1 and ω_2 fall such that $\omega_1 + \omega_2 = \omega_3$. For $\mu \ll 1$ it is .atural to present the field in the medium in the form (compare (2.32)):

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{e}_1 A_1(\mu z) e^{i(\omega_1 t - \mathbf{k}_1 \mathbf{r})} + \mathbf{e}_2 A_2(\mu z) e^{i(\omega_2 t - \mathbf{k}_2 \mathbf{r})} + \text{complex conjugate} \qquad (2.76)$$

Substituting (2.75)-(2.76) into equation

$$c^2 [\nabla [\nabla \mathbf{E}]] + \frac{\partial^2 \mathbf{D}}{\partial t^2} = 0, \text{ where } \mathbf{D} = \hat{\varepsilon}(t, \omega, z) \mathbf{E} \qquad (2.77)$$

and, using the procedure discussed in § 3 of this chapter, we arrive at the conclusion that the essential interaction between waves $E_{1,2}$ in the medium with a dielectric constant of the form (2.75) can take place only with the fulfillment of condition $k_1 + k_2 = k$ (compare (2.34)), where the process of the change in complex amplitudes A_1 and A_2 in the space is described by truncated equations of the form

$$k_1 \cos \hat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \hat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_1}{dz} + i\eta_1 \omega_1^2 A_2^* = 0; \qquad (2.78a)$$

$$k_2 \cos \hat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \hat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA_2}{dz} + i\eta_2 \omega_2^2 A_1^* = 0, \qquad (2.79b)$$

where

$$\eta_1 = \frac{1}{2c^2} (\mathbf{e}_1 \hat{\varepsilon}_1 \mathbf{e}_2); \quad \eta_2 = \frac{1}{2c^2} (\mathbf{e}_2 \hat{\varepsilon}_1 \mathbf{e}_1); \qquad (2.79)$$

It is easy to see that equations (2.78) have the same form as that of equations (2.60) and, consequently, allow the existence growing solutions of the form (2.62), which describe the amplification of waves $E_{1,2}$. The indicated intensification in terms of the three-frequency interaction of waves in a nonlinear medium should,

obviously, be treated as a forced coherent process of disintegration of photons of frequency ω_3, which occurs under action of photons of frequencies ω_1 and ω_2. On the other hand, with the use of concepts on the medium with variables parameters the process of amplification can be treated as the result of the work produced by the nonstationary medium above waves $E_{1,2}$ and the very amplification of the waves can be called parametric amplification.

Similarly, the interaction of waves described by equations (2.66)-(2.67) can be treated as the parametric interaction in the medium, the dielectric constant of which has the form

$$\hat{\varepsilon}(t, \omega, z) = \hat{\varepsilon}_0(\omega) + \mu \hat{\varepsilon}_1 \left\{ e^{i(\omega_2 t - k_2 z)} + e^{-i(\omega_2 t - k_2 z)} \right\}. \tag{2.80}$$

The process of spatial beats occurring between waves of frequencies ω_1 and ω_3 can be called the parametric conversion of frequency in a medium with variable parameters.

Of course, the consideration based on equations of the type (2.30) and their results (2.60) and (2.66)-(2.67) is fuller than the consideration founded on (2.75) and (2.80), inasmuch as in the first place, here it is possible to analyze the conditions of applicability of concepts on the medium with variable parameters and, secondly, directly calculate the characteristics of the tensor of the second class $\hat{\varepsilon}_1$, in terms of characteristics of the tensor of the nonlinear polarizability of the quadratic medium. Actually, in virtue of (2.38) and (2.79)

$$\mathbf{e}_1 \hat{\varepsilon}_1 \mathbf{e}_2 = 4\pi \mathbf{e}_1 \hat{\chi}^{\omega_3, -\omega_2} \mathbf{e}_3 \mathbf{e}_2; \; \mathbf{e}_2 \hat{\varepsilon}_1 \mathbf{e}_1 = 4\pi \mathbf{e}_2 \hat{\chi}^{\omega_3, -\omega_1} \mathbf{e}_3 \mathbf{e}_1. \tag{2.81}$$

Within bounds of the uniform problem, the model of the medium, with parameters variable according to the law of the traveling wave, is applicable only for the dispersive medium. Actually, the use of formulas (2.75) and (2.80) assumes the absence of considerable distortions of the intense wave (frequencies ω_3 or ω_2) in the examined medium. The latter can take place if, first, waves at frequencies ω_1 and ω_2 or ω_1 and ω_3 can be examined as weak, and

excitation of harmonics $n\omega_3$, or $n\omega_2$ can automatically be disregarded. It is interesting that if the determining modulation of parameters of the medium is an intense force (or, as it is accepted to call pumping it, wave) can be essentially distorted, the value of the parametric amplification in such a medium does not exceed e times, inasmuch as precisely on the characteristic length of the parametric amplification $L_0^{(2)}$ (see formulas (2.63) and (2.73)) the sinusoidal wave of pumping is turned into a wave of the sawtooth form [compare (2.73) with (I.24)].[1]

Although the energy relations (2.43)-(2.44) and (2.57)-(2.58) by themselves do not give information about the "direction" of the nonlinear process(merging or disintegration) in the quadratic medium, in those cases when this information can be obtained from conditions (as takes place in the approximation of the assigned field) the energy relations permit estimating the effectiveness of the nonlinear interaction.

Let us note, first of all, that inasmuch as in interactions of the examined type there is always preserved the number of quanta, and the conversion of frequency "upwards" with an interaction of waves in a nonlinear nondissipative medium occurs considerably more effective than the conversion of frequency "downwards."

Let us turn, for example, to the problem on the generation of difference frequencies in a medium with quadratic polarization, which corresponds to boundary conditions (2.55). From (2.57) it follows that signs of increases in energy of waves at frequencies ω_1, ω_2, and $\Delta W_{1,2}$ on the segment of the nonlinear medium Δz are identical, and signs of increases ΔW_1 and ΔW_3 (and consequently, ΔW_2 and ΔW_3) are opposite. Four boundary conditions (2.55) and small z, $\Delta W_{1,2}>0$ and $\Delta W_3<0$; the latter means that independently of the relationship of the number of photons $N_2(0)$ and $N_3(0)$ the energy of the wave on freqauncy ω_1 with growth z increases only owing to the one most high-frequency wave. Here the energy removed

[1]The last circumstance is one of main difficulties standing in the way of the realization of acoustic parametric amplifiers of a traveling wave.

from the wave of frequency ω_3 is divided between waves of frequencies ω_2 and ω_1 with respect to

$$\frac{\Delta W_1}{\Delta W_2}=\frac{\omega_1}{\omega_2} \qquad (2.82)$$

and when $\frac{\omega_1}{\omega_2}\ll 1$ the increase in energy of the low-frequency wave is small.[1] Conversely, in the problem on the generation of sum frequencies, for boundary conditions of the type (2.54), $\Delta W_{1,2}<0$ and $\Delta W_3>0$ and, consequently, all the energy of low-frequency photons participating in the nonlinear interaction passes into a high-frequency wave.

Relations analogous to those above can be used in the analysis of general regularities of parametric amplification and conversion of frequency. In particular, formula (2.82) describes, obviously, the relationship of increases in energies of growing waves in the problem on the parametric amplification.

In conclusion of this point, let us note that although the classification of nonlinear boundary value problems given above pertained to three-frequency interactions in the quadratic medium, analogous considerations can be assumed as the basis of the classification of different boundary value problems capable of appearing within the bounds of four-frequency interactions (cubic medium). Thus, just as in the three-frequency interaction, in the four-frequency interaction, in general, there simultaneously occur processes of disintegration and merging of photons, (complicated by effects of the "self-action" of the waves, see formula (1.32)). For example, the second equation of (2.49) describes the process of merging of three photons of frequency ω, and the first equation of (2.49) – the reverse process of disintegration of photons of frequency 3ω.

When $N_1(0)\gg N_2(0)$ in any case for border the process of merging

[1]Let us stress that this conclusion pertains only to nonlinear interactions in a nondissipative medium (medium with "reactive" nonlinearity). However, an effective generation of waves of difference frequencies proves to be possible in media with dissipative nonlinearity, for which energy relations of the type (2.57) are already inapplicable (see [95]).

of photons (the process of generation of the third harmonic) predominates.

For a cubic medium the spatial scale of nonlinear four-frequency interactions $L_0^{(3)}$ has the form [compare equations (2.39) and (2.49) and also formula (2.73)]:

$$L_0^{(3)} \simeq \frac{1}{\theta A_n^2(0)\, k_n}. \tag{2.83}$$

For $l_\kappa < L_0^{(3)}$, just as in the fulfillment of condition (2.74) in the quadratic medium, in the whole cubic medium it is possible to consider only that nonlinear process (merging or disintegration) which plays the determining role on its border. Condition $l_\kappa < L_0^{(3)}$, is thus the condition of applicability of the method of the assigned field in the analysis of four-frequency interactions. The effectiveness of a certain four-frequency interaction, just as the three-frequency can be estimated with the help of energy relations of the type (2.51)-(2.53).

4.2. Side Forces in a Nonlinear Medium. Truncated Equations of a Nonuniform Problem of Electrodynamics of a Nonlinear Medium

The method of conclusion of truncated equations, discussed in § 3 of this chapter can easily be generalized in the case of a nonuniform problem.

As an example let us examine the three-frequency interaction in a quadratic medium, at each point of which there acts a side force — side current with desnity I(t, r).

Then initial equations of the nonuniform problem have the form [compare (I.1)]

$$\mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \tag{2.84a}$$

$$\mathrm{rot}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial \mathbf{P}}{\partial t} + \frac{4\pi}{c}\mathbf{I}(t, \mathbf{r}), \tag{2.84b}$$

and the coupling of the vector of polarization P with field E is recorded in accordance with results of Chapter I. The second order equation, which corresponds to (2.2), for the nonuniform problem has the form:

$$[\nabla[\nabla \mathbf{E}]] + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}^{(2)}}{\partial t^2} + \mu F\left(t, \mathbf{r}, \mathbf{E}, \frac{\partial^2 \mathbf{E}}{\partial t^2}\right) + \frac{4\pi}{c^2}\frac{\partial \mathbf{I}}{\partial t} = 0. \qquad (2.85)$$

The method of deriving equations of the first approximation corresponding to equation (2.85) is analogous to that discussed in § 2-3. At first it follows to find the general solution in the zero approximation ($\mu = 0$) and then to clarify how nonlinear and dissipative terms disturb. Here the question of the selection of the order of smallness of the side current is very important which should be determined from physical considerations.

In many problems of nonlinear optics the appearance of side forces in Maxwell equations is connected with natural fluctuations in the medium. In this case it is natural to consider $I \sim \mu$ (or even $I \sim \mu^2$) and therefore to examine the plane monochromatic waves of constant amplitude as natural waves of the medium in a zero approximation.

Then the solution of (2.85) for the case of a three-frequency interaction can be sought in the form of the superposition of three waves with complex amplitudes (2.32) slowly variable in time and in space. It is necessary to stress that in the nonuniform problem the dependence of the complex amplitudes of waves in the medium on time, in general, takes place also when waves falling on a nonlinear medium, are unmodulated; the latter is connected with the time dependence of the side current I.

Let us assume that at first $I \sim \mu$. In this case the presence of the side force will already have an effect on the form of equations of the first approximation. Substituting (2.32) into (2.85), using (2.33), (2.35) and (2.36), multiplying in turn the obtained expression by

$$\mathbf{e}_1 \exp i(\omega_1 t - \mathbf{k}_1 \mathbf{r});\ \mathbf{e}_2 \exp i(\omega_2 t - \mathbf{k}_2 \mathbf{r});\ \mathbf{e}_3 \exp i(\omega_3 t - \mathbf{k}_3 \mathbf{r})$$

and, each time conducting term by term averaging over the period $T_{1,2,3}=\frac{2\pi}{\omega_{1,2,3}}$, we arrive at truncated equations of the nonuniform problem.

For amplitude A_1, for example, we have:

$$[\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\,\mathbf{s}_1\frac{\partial A_1}{\partial t}+[\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\,\nabla A_1+$$

$$+(\mathbf{e}_1\hat{\alpha}_1\mathbf{e}_1)\,\hat{A}_1+i\beta\omega_1^2 e^{+i\Delta r}A_3A_2^*+\frac{2\pi}{c^2}I_1(\mu t,\mu\mathbf{r})=0. \qquad (2.86)$$

Here

$$I_1(\mu t,\mu\mathbf{r})=\frac{\omega_1}{\pi}\int_{-\frac{\pi}{\omega_1}}^{\frac{\pi}{\omega_1}}\mathbf{e}_1\frac{\partial I}{\partial t}\exp i\omega_1 y\cdot dy. \qquad (2.87)$$

Quite similarly, in equations for A_2 and A_3 there appear terms $I_2(\mu t,\ \mu\mathbf{r})$, and $I_3(\mu t,\ \mu\mathbf{r})$, which can be obtained from (2.87) of the corresponding replacement of indices. Thus, the presence of a side current distributed over the medium in the first approximation leads to the appearance of external forces acting on the slowly changing complex amplitudes of interacting waves. Formula (2.87) shows that the essentially the flow of a three-frequency interaction in a quadratic medium is affected only by those components of the side current which can be represented in the form of the superposition of three waves similar in their structure to waves of (2.32), i.e., for an analysis of the influence of side force $\dot{\mathbf{I}}=\frac{\partial I}{\partial t}$ entering into (2.85) on the process of the three-frequency interaction one should separate from $\dot{\mathbf{I}}$ only components of the form

$$\dot{\mathbf{I}}=\sum_{n=1}^{3}\mathbf{e}_n I_n(\mu t,\mu\mathbf{r})\cdot\exp j(\omega_n t-\mathbf{k}_n\mathbf{r}). \qquad (2.88)$$

If current $I(t,\ r)$ is random, the slowly changing functions I_n are random. Statistical characteristics of the latter can be easily determined if statistical characteristics of the random field I are known.

Let us note that the fruitfulness of concepts on side fluctuating forces in the theory of natural fluctuations of a nonquasi-stationary linear medium was first demonstrated in the monograph

of S. M. Rytov [108]. There is the possibility of using these results in the investigation of statistical phenomena in a nonlinear medium. Certain concrete results obtained in this direction are discussed in work [56].

In the case, when according to conditions of the problem there should be ascribed the second order of smallness ($I \sim \mu^2$) to the outside force, the complex amplitudes can be represented in the form:

$$A_n(\mu t, \mu \mathbf{r}) = A_n^{(0)}(\mu t, \mu \mathbf{r}) + \mu a_n(\mu t, \mu \mathbf{r}), \tag{2.89}$$

where small additions of a_n characterize the change in the complex amplitudes due to the influence of side forces. Here if on the border of the nonlinear medium waves E_n are unmodulated, instead of (2.89) it is possible to write:

$$A_n(\mu t, \mu \mathbf{r}) = A_n^{(0)}(\mu \mathbf{r}) + \mu a_n(\mu t, \mu \mathbf{r}). \tag{2.90}$$

Equations for $A_n^{(0)}$ have the same form as those for the uniform problem, and equations for a_n have the structure of equations (2.86). Thus, for a_1, for example, we have [compare (2.86)]:

$$\begin{aligned}&[\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\,\mathbf{s}_1\frac{\partial a_1}{\partial t} + [\mathbf{e}_1[\mathbf{k}_1\mathbf{e}_1]]\,\nabla a_1 + (\mathbf{e}_1\hat{\alpha}\,\mathbf{e}_1)\,a_1 + \\ &+ i\beta\omega_1^2 e^{+i\Delta\mathbf{r}}\cdot(A_3^{(0)}a_2^* + A_2^{(0)*}a_3) + \frac{2\pi}{c^2}I_1(\mu t, \mu\mathbf{r}) = 0\,.^{1}\end{aligned} \tag{2.91}$$

Thus, the appearance of small ($\sim\mu$ or μ^2) side forces does not change the general form of the solution of (2.32) and only changes by in some measure the behavior of the slowly changing complex amplitudes. With a sufficient degree of accuracy one can assume that here, just as in the uniform problem, the "direction" of the nonlinear process (merging or disintegration) in any case near the border of the medium is determined by boundary conditions.

The presence of intensive side forces ($I \sim \mu^0$) is reflected already in the form of resolution of the problem obtained in zero ($\mu = 0$) approximation. Actually, here the equation of zero approximation should be recorded in the form (compare (2.5))

[1]For simplicity here we do not consider other effects having an order of μ^2.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + 4\pi \frac{\partial^2 \mathbf{p}^{(s)}}{\partial t^2} + 4\pi \frac{\partial \mathbf{I}}{\partial t} + c^2 [\nabla [\nabla \mathbf{E}]] = 0. \tag{2.92}$$

From (2.92) forced waves $E^{(B)}$ excited in the medium by side forces can be determined. Let us assume that, for example,

$$\frac{\partial \mathbf{I}}{\partial t} = \mathbf{m} I_0 \exp i(\omega t - \mathbf{k}_I \mathbf{r}), \tag{2.93}$$

where in general $k_I m \neq 0$; the side force is not required to have the form of a transverse wave.

The forced wave can be written in the form

$$\mathbf{E}^{(B)} = \mathbf{A} \exp i(\omega t - \mathbf{k}_I \mathbf{r}) \tag{2.94}$$

and, consequently, the amplitude of the forced wave is determined by the relation (compare (2.8))

$$\omega^2 \mathbf{A} + 4\pi\omega^2 \hat{\varkappa}(\omega) \mathbf{A} + c^2 [\mathbf{k}_I [\mathbf{k}_I \mathbf{A}]] + 4\pi \mathbf{m} I_0 = 0. \tag{2.95}$$

For the isotropic medium, and also for the case when in the anisotropic medium the optical axis is perpendicular to the plane k_I, m, it is possible to introduce the scalar dielectric constant $\varepsilon(\omega) = 1 + 4\pi\kappa(\omega)$, and then for $k_I m = 0$ we have:

$$\mathbf{E}^{(B)} = \frac{4\pi\omega^2 I_0}{c^2 (k_I^2 - k_\omega^2)} \cdot \mathbf{m} \exp i(\omega t - \mathbf{k}_I \mathbf{r}), \tag{2.96}$$

where $k_\omega = \frac{\omega}{c} \sqrt{\varepsilon(\omega)}$.

From (2.96) it follows that for the weakly absorbing medium ($Im\kappa \sim \mu$), the assignment of side forces in a zero approximation is correct only in the case when they do not have a resonance action on the medium, i.e., if the phase speeds of forced waves are not equal to phase speeds of natural waves of the medium at corresponding frequencies (for a side force of the form (2.93) the field of the forced wave is finite if $k_I \neq k_\omega$).

In a quadratic medium field $E^{(B)}$ excites the wave of polarization at frequency 2ω:

$$\mathbf{P}^{2\omega} = \mathbf{p}\chi^{2\omega}[A^{(B)}]^2 \exp i(2\omega t - 2\mathbf{k}_I \mathbf{r}). \tag{2.97}$$

The amplitude of the wave of the second harmonic, excited in the medium of the wave polarization (2.97) will be finite if $k_{2\omega} \neq 2k_I$ (see formula (I.21)-(I.22)).

Thus, in a nonlinear medium the forced wave $E^{(B)}$ can be examined as the assigned only until distortions of its form are stored. Everything that has been said means that concepts on side forces naving an order of μ_0 in the examined theory does not lead to internal contradictions only when forced waves, excited by these forces, can be examined as stationary, which do not undergo noticeable distortions in the medium. It is natural, therefore, to treat the influence of such waves on the medium as the modulation of its parameters. Let us discuss this question in somewhat greater detail.

4.3. Media with Variables Parameters

In virtue of (I.6), the vector of nonlinear polarization of the quadratic medium, which is under the influence of a strong side field ($I \sim \mu^0$) and natural ("free") wave E

$$\begin{aligned} \mathbf{P}^{(2)} = \int_0^\infty dt' \int_0^\infty \hat{\chi}(t', t'') \mathbf{E}(t-t') \mathbf{E}^{(B)}(t-t'-t'') dt'' + \\ + \int_0^\infty dt' \int_0^\infty \hat{\chi}(t', t'') \mathbf{E}(t-t') \mathbf{E}(t-t'-t'') dt'' + \\ + \int_0^\infty dt' \int_0^\infty \hat{\chi}(t', t'') \mathbf{E}^{(B)}(t-t') \mathbf{E}^{(B)}(t-t'-t'') dt''. \end{aligned} \tag{2.98}$$

(Here reduced symbolic notation is used.) The first term in (2.98) describes the interaction of waves E and $E^{(B)}$, the second – the distortion of wave E and the third – distortion of wave $E^{(B)}$. If $|\mathbf{E}^{(B)}| \gg |\mathbf{E}|$, with substitution of (2.98) into the Maxwell equations, it is possible to hold only the first term proportional to $E \cdot E^{(B)}$ (stored distortions of wave $E^{(B)}$ are impossible, and distortions of wave E against the background of the influence of wave $E^{(B)}$ on E are unobtrusive). In this case the behavior of wave E in the quadratic

medium, which is under the influence of an intense side force, is described by the following equations [compare with equations (2.84)]:

$$\mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \tag{2.99a}$$

$$\mathrm{rot}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial}{\partial t}\int_0^\infty \hat{\varkappa}(\mathbf{r}, t, t')\mathbf{E}(t-t')\,dt', \tag{2.99b}$$

where, the polarizability $\kappa(r, t, t')$ dependent on time t and in general on coordinate r, yields expression [see (2.98)]:

$$\hat{\varkappa}(\mathbf{r}, t, t') = \hat{\varkappa}(t') + \int_0^\infty \hat{\chi}(t', t'')\,\mathbf{E}^{(0)}(r, t-t'-t'')\,dt'' = \\ = \hat{\varkappa}(t') + \hat{M}(\mathbf{r}, t, t'). \tag{2.100}$$

Equation (2.99) can be used in the investigation of the propagation of relatively weak waves in a cubic medium, which is under the influence of intense side electromagnetic fields.

In this case, the polarizability of the equivalent medium with variable parameters can be represented in the form

$$\hat{\varkappa}(\mathbf{r}, t, t') = \hat{\varkappa}(t') + \hat{M}(\mathbf{r}, t, t'), \tag{2.101}$$

where

$$\hat{M}(\mathbf{r}, t, t') = \int_0^\infty dt'' \int_0^\infty dt''' \hat{\theta}(t', t'', t''')\mathbf{E}^{(0)}(\mathbf{r}, t-t'-t'')\times \\ \times \mathbf{E}^{(0)}(\mathbf{r}, t-t'-t''-t'''). \tag{2.102}$$

If the modulation of parameters of the medium is produced by a side electrical field, properties of the symmetry of a tensor of the second order M_{mn} are directly determined by properties of the symmetry of tensors $\hat{\chi}$ or $\hat{\theta}$, which is investigated in Chapter I in detail. In spite of the fact that in majority of practically interesting problems the model of the medium with variable parameters appears [see, for example (2.100) and (2.101)] as the maximum case of the general problem on the interaction of oscillations and waves in a nonlinear medium,[1] the approach based on direct

[1]Comparatively slow changes of properties of the medium can be obtained with the influence of forces of nonelectric origin. A classical example is the modulation of the dielectric constant of the medium with the help of ultrasonic waves. Detailed theoretical research of the propagation of light in such a medium is given in the work of S. M. Rytov [109].

consideration of equations of the type (2.99) is of interest. The theory in which equations containing tensor $M_{mn}(r, t, t')$, appear as the initial can be called the electrodynamics of nonstationary media. The expediency of construction of such a theory is connected to a considerable degree with the importance of clarification on its basis of general properties of quantum-mechanical and parametric amplifiers. The nonequilibrium of the medium, utilized in such amplifiers, is conditioned, as is known, by a variable external influence – oscillations or pumping waves. Examination of the general theory of media with variable parameters emerges beyond the framework of this book; we refer the reader to a number of works of F. V. Bunkin and colleagues [117]-[124] in which for periodically nonstationary medium general properties of tensor $\hat{M}$, radiation, natural fluctuations etc., are investigated. In this chapter we will limit ourselves to certain remarks referring to the propagation of waves in a medium with variable parameters. It is expedient to distinguish here two groups of problems:

I. Problems connected with the propagation of waves in a medium whose parameters are changed only with time (nonstationary spatially homogeneous medium).

II. Problems connected with the propagation of waves in a medium whose parameters are changed both with time and space (nonstationary spatially nonuniform medium). Although, in principle, in both of the indicated cases there is interest in the arbitrary law of the change in parameters of the medium, experimentally realized situations of the case when parameters of the medium are changed periodically correspond most closely.

Being interested here only in the fundamental side of matter, we will consider for simplicity that the law of the change in parameters of the medium is the harmonic law. Then for the spatially homogeneous medium the vector of polarization can be represented in the form

$$\mathbf{P} = \int_{0}^{\infty} \hat{\varkappa}(t')\mathbf{E}(t-t')\,dt' + e^{i\Omega t}\int_{0}^{\infty} \hat{M}(t')\mathbf{E}(t-t')\,dt' \qquad (2.103)$$

+ complex conjugate,

(where Ω – frequency of the change in the parameter), and for a spatially nonuniform nonstationary medium

$$\mathbf{P} = \int_0^\infty \hat{\varkappa}(t')\mathbf{E}(t-t')dt' + e^{i(\Omega t - \mathbf{k}_\Omega \mathbf{r})}\int_0^\infty \hat{M}(t')\mathbf{E}(t-t')dt' + \text{complex conjugate.} \tag{2.103a}$$

Here k_Ω – wave vector of the wave of the change in the parameter.[1] Equations of electrodynamics of media variable parameters are simpler than corresponding equations of a nonlinear medium. Therefore here, in any case for the periodically nonstationary medi, it is possible to record the form of the field in the medium. Moreover, exact solutions of equations of the type (2.99) can be obtained for certain, indeed rather artificially selected, nonperiodic laws of the change in properties of the medium (see, for example, works [112-113], where there is investigated the change in the amplitude and frequency of the electromagnetic wave propagating in the medium the parameters of which linearly or quadratically depend on time).

Let us assume that on the medium, the properties of which are described by formula (2.103), there falls a plane monochromatic wave of frequency ω. Then the general form of the plane wave in

---

[1]If modulation of parameters of the medium is carried out by the electrical field, using (2.100) and (2.101), one can determine the frequency Ω and characteristics of the tensor $\hat{M}$ according to the assigned polarization and frequency ω of the side field. For the quadratic medium, obviously $\Omega = \omega;\ \mathbf{k}_\Omega = \mathbf{k}_l$

$$\hat{M}(t') = \int_0^\infty \hat{\chi}(t', t'')\mathbf{A}^{(\mathrm{s})} \exp\{-i\Omega(t' + t'')\}\, dt''.$$

For a cubic medium $\Omega = 2\omega,\ \mathbf{k}_\Omega = 2\mathbf{k}_l$ and

$$\hat{M}(t') = \int_0^\infty \hat{\theta}(t', t'', t''')\, \mathbf{A}^{(\mathrm{s})}\, \mathbf{A}^{(\mathrm{s})} \exp\{-i\Omega(2t' + 2t'' + t''')\} dt'' dt'''.$$

Let us note also that with transition from a cubic medium to a model of the medium by variable parameters, in contrast to the case of the quadratic medium, the stationary part of the polarizability should be modified

$$\hat{\varkappa}(t') = \hat{\varkappa}^{(\mathrm{s})}(t') + \int_0^\infty\int_0^\infty \theta(t', t'', t''')\, \mathbf{E}^{(\mathrm{s})}(t - t' - t'')\, \mathbf{E}^{(\mathrm{s})*}(t - t' - t'' - t''')\, dt'' dt'''$$

in such a medium is given by the formula

$$\mathbf{E}(r,t)=\sum_{n=-\infty}^{n=+\infty}\mathbf{B}^{(n)}(\mathbf{r})\exp i(\omega+n\Omega)t. ^1 \qquad (2.104)$$

The fact that (2.104) indeed gives the general form of the wave in a periodically nonstationary medium it is easy to be convinced, using, for example, the method of successive approximations. (In virtue of (2.103) the wave of the frequency ω excites in a nonstationary medium waves with frequencies $\omega \pm \Omega$, and these waves in turn excite waves with frequencies $\omega \pm 2\Omega$ etc.).

If all waves, (2.104), are equivalent, substitution of (2.104) into Maxwell equations leads to an infinite system of differential second order equations for complex amplitudes $B^{(n)}(r)$. However, in concrete problems it is frequently not necessary to retain all waves in the solution of (2.104). An especially fruitful means of the simplification of the problem here, just as in the nonlinear problem, is the preliminary estimate of the order of smallness of different terms in (2.103). In many cases quantity $\hat{M}$ can be examined as small $\hat{M}\sim\mu$. Then, in the solution of the boundary value problem of the electrodynamics of a nonstationary medium, in any case for not too large z ($z = 0$, as earlier, corresponds to the boundary) in (2.104) it is possible to retain only the first two combination frequencies ($\omega + \Omega$; $\omega - \Omega$) and reject the others (their amplitudes have an order of μ^2, μ^3 and etc.).

Then the solution can be presented in the form

$$\mathbf{E}(\mathbf{r},t)=\sum_{n=-m}^{m}\mathbf{B}^{(n)}(\mu z,\mathbf{r})\exp i(\omega+n\Omega)t \qquad (2.105)$$

(here m no longer exceeds unity and two), and further we can use the method of derivation and analysis of truncated equations discussed in § 3 of this chapter. It is not difficult to be convinced here that in a spatial uniform periodically nonstationary medium, the obtaining of stored effects (monotonic change in complex overall amplitudes) in general is impossible.

[1]Complex amplitude $B^{(n)}$ contains the phase constant of the wave.

Actually, in accordance with (2.103), in a periodically nonstationary medium the wave of the field of the form

$$\mathbf{E}^{(n)} = \mathbf{A}^{(n)} \exp i\,[(\omega + n\Omega)t - \mathbf{k}_n \mathbf{r}], \qquad (2.106)$$

besides the wave of polarization at frequency $\omega_n = \omega + n\Omega$, also excites the wave of polarization at combination frequencies. These waves have the form:

$$\mathbf{P}^{\omega+\Omega(n+1)} = \hat{M}(\omega + n\Omega)\mathbf{A}^{(n)} \exp i\,[(\omega + n\Omega + \Omega)t - \mathbf{k}_n \mathbf{r}] + \\ + \text{complex conjugate} \qquad (2.107)$$

$$\mathbf{P}^{\omega+\Omega(n-1)} = \hat{M}(\omega + n\Omega)\mathbf{A}^{(n)} \exp i\,[\omega + n\Omega - \Omega)t - \mathbf{k}_n \mathbf{r}] + \\ + \text{complex conjugate} \qquad (2.108)$$

Here $\hat{M}(\omega + n\Omega) = \int_0^\infty \hat{M}(t')e^{-i(\omega+n\Omega)t'}dt'$ – Fourier-component of the tensor $\hat{M}$.

From (2.107) and (2.108) it follows that neither in the medium without dispersion nor in the medium with normal dispersion do forced waves of polarization have a resonance effect on the medium. Actually, for both of the indicated cases cannot be simultaneously fulfilled the relations $\mathbf{k}_{n-1} = \mathbf{k}_n$; $\mathbf{k}_{n+1} = \mathbf{k}_n$. Therefore, truncated equations for slowly changing amplitudes will contain in the right sides oscillatory terms of the form:

$$\frac{2\pi}{c^2}\, \mathbf{e}_n \hat{M} \mathbf{e}_{n-1} \mathbf{A}^{(n-1)} \exp i\,[(\mathbf{k}_{n-1} - \mathbf{k}_n)\mathbf{r}]$$

(e_n – unit vector characterizing, as earlier, polarization of the wave $E^{(n)}$), the presence of which prevents the appearance of stored effects. In particularity, in a sufficiently extended peri ically nonstationary medium, even for $\Omega > \omega$ parametric amplification proves to be impossible (in application to the periodically transient plasma this is shown in works [122]-[123]).

Another situation takes place in the case when the periodically nonstationary medium is simultaneously and spatially nonuniform and is described by formula (2.103a). In this case instead of (2.107) and (2.108) we have:

$$P^{\omega+\Omega(n+1)} = \hat{M}(\omega + n\Omega)\,\mathbf{A}^{(n)} \exp i\left[(\omega + n\Omega + \Omega)t - \right.$$

$$\left. -(\mathbf{k}_\Omega + \mathbf{k}_n)\mathbf{r}\right] + \text{complex conjugate} \qquad (2.109)$$

$$P^{\omega+\Omega(n-1)} = \hat{M}(\omega + n\Omega)\,\mathbf{A}^{(n)} \exp i\left[(\omega + n\Omega - \Omega)t - \right.$$

$$\left. -(\mathbf{k}_n - \mathbf{k}_\Omega)\mathbf{r}\right] + \text{complex conjugate} \qquad (2.110)$$

If $\mathbf{k}_\Omega + \mathbf{k}_n = \mathbf{k}_{n+1}$; $\mathbf{k}_n - \mathbf{k}_\Omega = \mathbf{k}_{n-1}$ waves of polarization (2.109) and (2.110) have on the medium a resonance effect and thus can lead to stored effects.

A special case of such resonance interaction is the case of parametric amplification of two waves examined above with frequencies ω_1 and ω_2, which satisfy the relationship $\omega_1 + \omega_2 = \omega_3$ (in designations of formula (2.104) $\omega_1 = \omega$; $\omega_3 = \Omega$). At the same time in a medium with parameters variable in accordance with (2.103a) more complex multiwave interactions are possible. The general form of the plane wave in the medium whose properties are characterized by (2.103a) is given by formula [compare (2.104)]:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{n=-\infty}^{n=+\infty} \mathbf{B}^{(n)}(\mathbf{r}) \exp i\left[(\omega + n\Omega)t - n\mathbf{k}_\Omega \mathbf{r}\right]. \qquad (2.111)$$

In certain cases instead of (2.111) another notation at which proves to be more convenient (the complex amplitude is in the form $\mathbf{B}^{(n)}(\mathbf{r}) = \mathbf{C}^{(n)} e^{-i\Gamma \mathbf{r}}$, where $C^{(n)}$ is the vector constant. Then instead of (2.111) it is possible to write

$$\mathbf{E}(\mathbf{r}, t) = \exp i\left[\omega t - \Gamma \mathbf{r}\right] \sum_{n=-\infty}^{n=+\infty} \mathbf{C}^{(n)} \exp in\left[\Omega t - \mathbf{k}_\Omega \mathbf{r}\right]. \qquad (2.112)$$

The last expression is the result of Floquet theorem [125]. Actually, in accordance with the Floquet theorem

$$\mathbf{E}(\mathbf{r}, t) = \exp i\left[\omega t - \Gamma \mathbf{r}\right] \cdot \Phi(\Omega t - \mathbf{k}_\Omega \mathbf{r}).$$

and the sum in (2.112) corresponds to the expansion of function Φ in a series along space harmonics.

The general solution (2.112) should be substituted into the

Maxwell equation; then from the condition of the difference from zero of constants $C^{(n)}$ there can be obtained the dispersion equation, which connects quantities Γ and ω.

Directing the wave vector k_Ω along the z axis and assigning the specific form of polarization of the incident wave, using (2.103a) and equations (2.99), we can record for the nondispersive medium:

$$c^2\nabla^2\mathbf{E} = \frac{\partial^2}{\partial t^2}\left\{\left[\varepsilon_0 + \varepsilon_1 e^{i(\Omega t - k_\Omega z)} + \varepsilon_1^* e^{-i(\Omega t - k_\Omega z)}\right]\mathbf{E}\right\}. \qquad (2.113)$$

Here ε_0 and ε_1 – scalars.

Substituting (2.112) into (2.113), we arrive at an infinite system of equations of the form

$$\left[\varepsilon_0 - \frac{c^2(\Gamma + nk_\Omega z_0)^2}{(\omega + n\Omega)^2}\right]C^{(n)} + \frac{\varepsilon_1}{2}\left[C^{(n-1)} + C^{(n+1)}\right] = 0. \qquad (2.114)$$

Discussion and analysis of the system of the type (2.114) is contained in [126-130]. A general investigation of the dispersion equation of the medium with variable parameters proves to be very difficult. In specific problems, however, considerably simplifications are possible which are based on the fact that the dispersion characteristic of the real medium allows an effective interaction of only a finite number of waves. Therefore, the infinite system (2.114) can be replaced by the truncated system; with this the order of the dispersion equation appears finite. Additional simplifications are obtained taking into account the order of smallness of modulation percentage of parameters of the medium (usually, $\varepsilon_1 \sim \mu$).

The character of stored effects appearing in the medium with variable parameters depends on the relationship between frequencies Ω and ω. If $\omega \gg \Omega$, the monotonic change in amplitudes should be treated, obviously, as a stored (with distance) effect of the modulation of the high-frequency wave of a periodically nonstationary medium (by the field of the low-frequency side wave).

An example of such a situation will be discussed in Chapter V. Let us note only that it is more convenient to describe the effect of modulation in a time and not spectral language. Therefore, here instead of (2.112) in certain cases it is expedient to look for the solution in the form of a wave with modulated amplitude; here instead of the system of truncated equations in ordinary derivatives, one should examine one truncated equation in partial derivatives. Cases $\omega \sim \Omega$ and $\Omega > \omega$ correspond to conditions of parametric amplification and conversion of the frequency. Here it is expedient to examine the system of truncated equations recorded for complex amplitudes of different spectral components. Of course, these truncated equations can be equations in partial derivatives; with such a position of things it is necessary to encounter in problems parametric amplification and conversion of the modulated signals (see Chapter V).

§ 5. Surface Nonlinear Interactions. Reflection of a Plane Electromagnetic Wave from the Border of the Nonlinear Medium

5.1. Formulation of the Problem

In the preceding paragraph, with the classification of nonlinear interactions, beforehand we were assigned values of amplitudes and directions of wave vectors of interacting waves on the border of the nonlinear medium. In reality, one should consider amplitudes and directions of wave vectors of waves falling on the border of the nonlinear medium to be assigned. Therefore, with strict setting, examination of interactions of waves in the nonlinear medium should be preceded by the investigation of regularities of the reflection and refraction of waves on its border. Here there appears, thus, the whole range of problems connected with the generalization of formulas of Fresnel on nonlinear media. It is important to stress here that inasmuch as on the border of the nonlinear medium the principle superposition is disrupted in the generalization of Fresnel formulas one should examine not only cases of the fall of monochromatic

waves but also the problem about the interaction of waves of different frequencies on the border of the nonlinear medium.

A strict solution of the indicated problems (we will subsequently call them problems of "surface nonlinear effects"), just as that which took place for problems on "volume" nonlinear interactions discussed in preceding paragraphs, proves to be very difficult. However, for the most practically interesting case of the weakly nonlinear and weakly absorbing medium, it is possible to develop a comparatively simple method of analysis of surface nonlinear phenomena, which are based on the approximation of the assigned field. Actually, nonlinear interactions, which determine regularities of the reflection and refraction of waves, occur, obviously, in a very thin boundary layer, the linear dimensions of which $l_{\Gamma p}$ have an order of thickness of several atomic layers and do not exceed, in any case the wavelengths. Therefore, in a weakly nonlinear medium reactions of harmonics and combination frequencies, which appear with surface interactions, on generating waves can deliberately be disregarded; the last circumstance is, as was already indicated, the initial point of the approximation of the assigned field.

Below we will illustrate, following basically work [131], the indicated method in the example of the problem on the incidence of a plane monochromatic wave on the surface of a weakly nonlinear quadratic medium.[1]

Let us consider the half-space filled by the nondissipative quadratic medium whose nonlinear properties are described by the tensor $\chi_{mnm'}\sim\mu$. Let us assume that the boundary coincides with

[1]Although, in principle, an analogous problem can be stated for the cubic medium, the case of the quadratic medium in the problem of surface interactions is the most important. The fact is that in the surface layer of the cubic medium the potential function is no longer symmetric, so that the polarizability of this layer is described no longer by an equation of the type (1.17a) but rather by an equation of the type (1.41a); the nonlinear polarizability of the surface layer of the cubic medium is close in their characteristics of such for a quadratic medium and is described by a tensor of the third, and not the fourth rank., (Experimental confirmation of this fact was obtained in [31]).

plane (x, y), and let us direct the z axis inside the nonlinear medium (II) (Fig. 2-2). Let us assume that from the linear isotropic medium (I) onto the boundary at angle $\theta_1^{(n)}$ to the normal a plane monochromatic wave is incident. The electrical and magnetic field of the incident wave have the form:

$$\mathbf{E}_1^{(n)} = \mathbf{e}_1^{(n)} A_1^{(n)} \cdot \exp i\,(\omega t - \mathbf{k}_1^{(n)} \mathbf{r}); \tag{2.115a}$$

$$H_1^{(n)} = \frac{c}{\omega}\left[\mathbf{k}_1^{(n)} \mathbf{e}_1^{(n)}\right] A_1^{(n)} \exp i\,(\omega t - \mathbf{k}_1^{(n)} \mathbf{r}). \tag{2.115b}$$

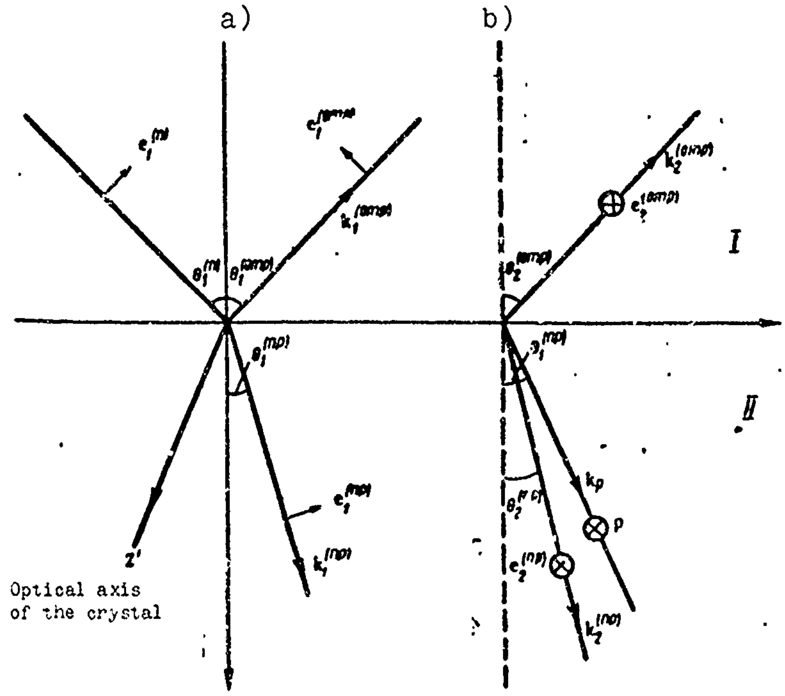

Fig. 2-2. Location of vectors in the problem on the reflection of a plane electromagnetic wave from the boundary of the quadratic medium: I – linear medium; II – nonlinear medium; ⊗ – denotes that the corresponding vector is directed along the normal to the plane of drawing; a) location of vectors for waves of frequency ω; b) location of vectors for waves of frequency 2ω, z' – edge of cubic crystal.

Subsequently we will be basically interested in surface nonlinear effects in optically isotropic crystals of the class T_d (precisely in such crystals these effects are studied experimentally in the most detail of all, see [202] and § 3 of Chapter VI). At the

same time, results of the conducted calculation, as one will see from subsequent computations, after a certain correction can be applied to anisotropic crystals of the class D_{2d}. In the last case we will assume that the direction of vector $\mathbf{e}_1^{(n)}$ is selected specially so that in the medium one refracted (extraordinary wave) was excited. We will consider further that the z' axis in Fig. 2-2 is directed along one of the edges of the cubic crystal (or along the optical axis of the uniaxial crystal).

With resolution of the problem on the reflection of the wave from the boundary of a weakly nonlinear medium, the sequence of calculation coincides with that accepted in §§ 1-3 of this chapter.

5.2. Zero Approximation. Reflection from the Border of the Linear Isotropic Medium

In zero $\mu = 0$ approximation the problem is reduced to the investigation of the reflection of the wave from the border of the linear medium. To solve it, linear Maxwell equations (I.1) and the condition of continuity of fields on the border should be used. For harmonic waves of the form (2.115) equations (I.1) can be recorded in the form:

$$\omega \mathbf{H} = c[\mathbf{kE}]; \quad \omega\varepsilon(\omega)\mathbf{E} = -c[\mathbf{kH}]. \qquad (2.116)$$

For a uniform medium without losses the reflected and passing waves are plane. Fields of passing and reflected waves can be presented in the form:

$$\mathbf{E}_1^{(\text{np})} = \mathbf{e}_1^{(\text{np})} A_1^{(\text{np})} \exp i(\omega t - \mathbf{k}_1^{(\text{np})}\mathbf{r}); \qquad (2.117)$$

$$\mathbf{E}_1^{(\text{otp})} = \mathbf{e}_1^{(\text{otp})} A_1^{(\text{otp})} \exp i(\omega t - \mathbf{k}_1^{(\text{otp})}\mathbf{r}). \qquad (2.118)$$

and corresponding magnetic fields can be recorded also just as in (2.115). All vectors $\mathbf{e}_1^{(n)}, \mathbf{e}_1^{(\text{np})}, \mathbf{e}_1^{(\text{otp})}$ lie in one plane (see Fig. 2-2). From the homogeneity of the problem in plane (x, y) it directly follows that:

$$k_{1x}^{(\text{п})} = k_{1x}^{(\text{пр})} = k_{1x}^{(\text{отр})} \tag{2.119}$$

and, consequently (all components $k_{1y}=0$)

$$\begin{gathered}\theta_1^{(\text{п})} = \theta_1^{(\text{отр})};\quad \frac{\sin\theta_1^{(\text{п})}}{\sin\theta_1^{(\text{пр})}} = \sqrt{\frac{\varepsilon_\parallel^{(II)}(\omega)}{\varepsilon_\parallel^{(I)}(\omega)}};\\ k_{1z}^{(\text{п})} = -k_{1z}^{(\text{отр})};\quad k_{1z}^{(\text{пр})} = \frac{\omega}{c}\sqrt{\varepsilon_\parallel^{(II)}(\omega) - \varepsilon_\parallel^{(I)}(\omega)\sin^2\theta_1^{(\text{п})}}.\end{gathered} \tag{2.120}$$

(Here $\varepsilon_\parallel^{(I)}(\omega)$ and $\varepsilon_\parallel^{(II)}(\omega)$ – spectral components of the linear dielectric constant of media I and II, which correspond to the assigned polarization $\mathbf{e}_1^{(\text{п})}$ parallel to the incidence plane). Relationships between the complex amplitudes of the incident, reflected and past waves can be set from conditions of continuity of tangential components of fields on the border. For the selected polarization of the incident wave these conditions should be recorded for components of the electrical field $E_{1x} = e_{1x}A$ and magnetic field $H_y = H = \frac{c}{\omega}kA$.

$$H|_{z=-0} = H|_{z=+0};\quad E_x|_{z=-0} = E_x|_{z=+0}. \tag{2.121}$$

Noting that in accordance with (2.116), $E_x = \frac{c}{\omega\cdot\varepsilon(\omega)}k_z H$, equalities (2.121) can be written in the form:

$$\left.\begin{gathered}H^{(\text{п})} + H^{(\text{отр})} = H^{(\text{пр})};\\ \frac{k_{1z}^{(\text{п})}}{\varepsilon_\parallel^{(I)}(\omega)}\cdot\left[H^{(\text{п})} - H^{(\text{отр})}\right] = \frac{k_{1z}^{(\text{пр})}}{\varepsilon_\parallel^{(II)}(\omega)}\cdot H^{(\text{пр})}.\end{gathered}\right\} \tag{2.122}$$

Solving equations (2.122), we arrive at Fresnel formulas for the wave of the chosen polarization

$$A_1^{(\text{пр})} = \frac{2\varepsilon_\parallel^{(II)}(\omega)k_{1z}}{\varepsilon_\parallel^{(I)}(\omega)k_{1z}^{(\text{пр})} + \varepsilon_\parallel^{(II)}(\omega)k_{1z}^{(\text{п})}}\cdot\sqrt{\frac{\varepsilon_\parallel^{(I)}(\omega)}{\varepsilon_\parallel^{(II)}(\omega)}}\,A_1^{(\text{п})}; \tag{2.123a}$$

$$A_1^{(\text{отр})} = \frac{\varepsilon_\parallel^{(II)}(\omega)\cdot k_{1z}^{(\text{п})} - \varepsilon_\parallel^{(I)}(\omega)\cdot k_{1z}^{(\text{пр})}}{\varepsilon_\parallel^{(I)}(\omega)k_{1z}^{(\text{пр})} + \varepsilon_\parallel^{(II)}(\omega)\cdot k_{1z}^{(\text{п})}}\cdot A_1^{(\text{п})}. \tag{2.123b}$$

From formulas (2.123) there can be calculated the linear reflectivity of the medium with respect to power, $R^{(\pi)}$:

$$R^{(\pi)} = \frac{\mathrm{tg}^2\left[\theta_1^{(np)} - \theta_1^{(n)}\right]}{\mathrm{tg}^2\left[\theta_1^{(np)} + \theta_1^{(n)}\right]}. \qquad (2.124)$$

Similarly there can be obtained formulas of Fresnel and for the wave polarized perpendicular to the plane of incidence. For the case of double refracting medium and geometry, selected in Fig. 2-2, formulas (2.120) and (2.124) should be corrected; here it is impossible to introduce $\varepsilon_t(\omega)$ (see [38]).

5.3. First Approximation. The Appearance of Harmonics in the Field of Reflected Wave

In a quadratic medium the passing wave (2.117) can be distorted, and, consequently, the spectrum of it can be enriched by harmonics, 2ω, 3ω, ... (the corresponding fields will be designated E_2, E_3 ...). The full field in the transition layer of a weakly nonlinear medium can obviously, be represented in the form:

$$\mathbf{E}^{(np)} = \mathbf{E}_1^{(np)} + \mu\mathbf{E}_2^{(np)} + \mu^2\mathbf{E}_3^{(np)} + \cdots \qquad (2.125)$$

It is necessary to stress that here, in contrast to the problem on volume nonlinear interactions, where we did not make, in general, assumptions on the smallness of fields of harmonics or combination frequencies but proceeded only from the slowness of the change in them in space, the actual fields of harmonics can be considered small. The growth of the order of smallness with the number of the harmonic for surface nonlinear interactions is natural, inasmuch as the m-harmonic appears in the quadratic medium as a result of the nonlinear interaction of the $(m - 1)$-harmonic with the field of basic radiation. In the first approximation $\mathbf{E}^{(np)} = \mathbf{E}_1^{(np)} + \mu\mathbf{E}_2^{(np)}$, and calculation of the characteristic of wave $\mathbf{E}_2^{(np)}$ can be conducted in the approximation of the assigned field with the help of the following equations:

$$\mathrm{rot}\,\mathbf{E}_2^{(np)} = -\frac{1}{c}\frac{\partial H_2^{(np)}}{\partial t}; \qquad (2.126a)$$

$$\mathrm{rot}\,H_2^{(np)} = \frac{\varepsilon(2\omega)}{c}\frac{\partial^2 \mathbf{E}_2^{(np)}}{\partial t^2} + \frac{4\pi}{c}\frac{\partial^2}{\partial t^2}\mathbf{P}^{2\omega}, \qquad (2.126b)$$

where

$$\mathbf{P}^{2\omega} = \hat{\chi}\mathbf{e}_1^{(np)}\cdot\mathbf{e}_1^{(np)}[A_1^{(np)}]^2 \exp i\,[2\omega t - \mathbf{k}_p\mathbf{r}] = = \mathbf{p}P^{2\omega}\exp i\,[2\omega t - \mathbf{k}_p\mathbf{r}]. \qquad (2.127)$$

The general solution of equations (2.126) is the superposition of two waves of the frequency 2ω – natural and forced [see (I.21), (I.22) and (2.96)], the wave vectors and polarization of which are determined by polarization p and wave vector k_p compelling wave of nonlinear polarization (2.127) and by boundary conditions.

In accordance with (1.45), $\mathbf{k}_p = 2\,\mathbf{k}_1^{(np)}$; the direction of vector p can be defined by well-known properties of the tensor of nonlinear polarizability $\chi_{mnm'}$ (see § 7 of Chapter 1) and assigned polarization of wave $E_1^{(np)}$. Inasmuch as the general properties of symmetry of the tensor $\chi_{mnm'}$ are determined by properties of symmetry of the crystal, along with the system of coordinates introduced in Fig. 2-2, it is expedient to introduce also one more system of Cartesian coordinates whose position of the axes is determined by the position of the axes of symmetry.

Let us consider as an example a crystal of the T_d type; as was already indicated in Chapter I, crystals of this type allow existence of tensor χ. Let us introduce the system of coordinates x', y' and z' connected with three axes of the cube. Noting that according to conditions of the problem vectors $\mathbf{z}'_0$, $\mathbf{k}_1^{(np)}$ and $\mathbf{e}_1^{(np)}$ lie in one plane for components of vector $\mathbf{e}_1^{(np)}$ along axes x', y' and z', we have:

$$e_{1,x'}^{(np)} = -\cos\theta\cos\varphi; \qquad (2.128a)$$

$$e_{1,y'}^{(np)} = -\cos\theta\sin\varphi; \qquad (2.128b)$$

$$e_{1,z'}^{(np)} = \sin\theta, \qquad (2.128c)$$

where θ — angle between the beam vector $\mathbf{s}_1^{(np)}$ and axis z', and ϕ the angle between projections of vector $\mathbf{s}_1^{np}$ on plane x' and y' counted off from the x' axis. In accordance with data of paragraph 20, § 7 Chapter I, for crystals of the class T_d only components $\chi_{x',y'z'}$, $\chi_{y',x'z'}$, and $\chi_{z',y'x'}$ are different from zero. Therefore, components of vector $P^{2\omega}$ in axes x', y' and z' are equal to:

$$P_{x'}^{(2\omega)} = -\frac{1}{2}\chi_{x',y'z'}^{2\omega}\cdot\left[A_1^{(np)}\right]^2 \sin 2\theta \cdot \sin\varphi; \tag{2.129a}$$

$$P_{y'}^{(2\omega)} = -\frac{1}{2}\chi_{y',x',z'}^{2\omega}\cdot\left[A_1^{(np)}\right]^2 \sin 2\theta \cos\varphi; \tag{2.129b}$$

$$P_{z'}^{(2\omega)} = \frac{1}{2}\chi_{z',y',x'}^{2\omega}\cdot\left[A_1^{(np)}\right]^2 \cos^2\theta \sin 2\varphi. \tag{2.129c}$$

Let us note that these components have the same form for a crystal of the D_{2d} type and geometry of Fig. 2-2.

Thus, vector $P^{2\omega}$ in the examined example has components which are both parallel and perpendicular to planes $\mathbf{z}_0$, $\mathbf{k}_1^{(np)}$. Let us assume that axis y' coincides with axis y $(\varphi=0)$. Then $p = y_0$, the wave of the nonlinear polarization, is transverse and excites waves in the medium at a frequency 2ω (see Fig. 2-2b).

Field $\mathbf{E}_2^{(np)}$ in the medium can be represented in the form:

$$\mathbf{E}_2^{(np)} = \mathbf{e}_2^{(np)} A_2^{(np)} \exp i\left(2\omega t - \mathbf{k}_2^{(np)}\mathbf{r}\right) + $$
$$+ \mathbf{y}_0 \cdot \frac{8\pi\omega^2}{c^2}\cdot\frac{\chi_{y',x',z'}^{2\omega}\cdot\sin 2\theta\cdot\left[A_1^{(np)}\right]^2}{k_p^2 - \left[k_2^{(np)}\right]^2}\exp i\left[2\omega t - \mathbf{k}_p\mathbf{r}\right], \tag{2.130a}$$

and the corresponding magnetic field:

$$H_2^{(np)} = \frac{c}{2\omega}\left[\mathbf{k}_2^{(np)}\mathbf{e}_2^{(np)}\right] A_2^{(np)} \exp i\left(2\omega t - \mathbf{k}_2^{(np)}\mathbf{r}\right) + $$
$$+ \left[\mathbf{y}_0\mathbf{k}_p\right]\frac{c}{2\omega}\cdot\frac{8\pi\omega^2}{c^2}\cdot\frac{\chi_{y',x',z'}^{2\omega}\cdot\sin 2\theta\cdot\left[A_1^{(np)}\right]^2 \exp i\left(2\omega t - \mathbf{k}_p\mathbf{r}\right)}{k_p^2 - \left[k_2^{(np)}\right]^2}. \tag{2.130b}$$

Waves (2.130) can satisfy the boundary conditions only in the case when in the vacuum there propagates plane wave of the form

$$E_2^{(\text{отр})} = e_2^{(\text{отр})} \cdot A_2^{(\text{отр})} \exp i(2\omega t - k_2^{(\text{отр})} r) \tag{2.131}$$

(where in virtue of the selection of angle ϕ made above, vectors $e_2^{(\text{отр})}$, $e_2^{\text{пр}}$ and y_0 are parallel). The latter means that the appearance of the wave of nonlinear polarization (2.127) in the medium should inevitably lead to excitation of the second harmonic not only in the field of the refracted wave but also in the field of the reflected wave.

The direction of wave vectors $k_2^{(\text{отр})}$ and $k_2^{(\text{пр})}$ and amplitudes of reflected and refracted waves of the second harmonic will be determined from boundary conditions. Thus, just as in the zero approximation (see (2.119)), for components of wave vectors along the x axis we have:

$$k_{2x}^{(\text{пр})} = k_{2x}^{(\text{отр})} = k_{px} = 2k_{1x}^{(\text{пр})} \tag{2.132}$$

(in accordance with the geometry selected in Fig. 2-2, all $k_{2y} = 0$).

Using (2.132), one can determine angles $\theta_2^{(\text{пр})}$ and $\theta_2^{(\text{отр})}$. Considering (2.120), we have:

$$\frac{\sin\theta_2^{(\text{пр})}}{\sin\theta_1^{(\text{пр})}} = \frac{\sqrt{\varepsilon_\parallel^{(II)}(\omega)}}{\sqrt{\varepsilon_\perp^{(II)}(2\omega)}}; \tag{2.133}$$

$$\sin\theta_2^{(\text{отр})} = \sqrt{\frac{\varepsilon_\parallel^{(I)}(\omega)}{\varepsilon_\perp^{(I)}(2\omega)}} \cdot \sin\theta_1^{(\text{п})}, \tag{2.134}$$

From (2.133) it follows that in general the direction of the wave vector of the natural wave of the second harmonic in a nonlinear medium differs from the direction of the wave vector of the refracted wave of the main frequency and, consequently, and wave of nonlinear polarization; $\theta_1^{(\text{пр})} = \theta_2^{(\text{пр})}$ only when $\varepsilon_\perp^{(II)}(2\omega) = \varepsilon_\parallel^{(II)}(\omega)$. The

relationship between the angle of incidence $\theta_1^{(n)}$ and the angle $\theta_2^{(отр)}$ characterizing the direction of the wave vector of the second harmonic, radiated in the linear medium, depends, as follows from (2.134), on the dispersion properties of the latter. In particular, if the linear medium is nondispersive, the "reflected wave" of the second harmonic propagates in the same direction as does the reflected wave of the main frequency.

For waves polarized perpendicular to the plane of incidence, the relationships between amplitudes can be established from the condition of continuity on the boundary of components $E_y = E$ and H_x.

$$E|_{z=+0} = E|_{z=-0}; \quad H_x|_{z=+0} = H_x|_{z=-0}. \tag{2.135}$$

From the first condition of (2.135), designating $P_y^{2\omega} = \frac{1}{2}\chi^{2\omega}_{y'x'z'}\sin 2\theta \times [A_1^{(np)}]^2$, we have:

$$A_2^{(отр)} = A_2^{(np)} + \frac{4\pi P_y^{2\omega}}{\varepsilon_\perp^{(II)}(2\omega) - \varepsilon_\parallel^{(II)}(\omega)}. \tag{2.136}$$

From the second condition of (2.135), using relation (2.116), we have:

$$-A_2^{(отр)}\sqrt{\varepsilon_\perp^{(I)}(2\omega)}\cos\theta_2^{(отр)} = A_2^{(np)}\sqrt{\varepsilon_\perp^{(II)}(2\omega)}\cos\theta_2^{(np)} + \\ + \frac{4\pi P_y^{2\omega}}{\varepsilon_\perp^{(II)}(2\omega) - \varepsilon_\parallel^{(II)}(\omega)}\cdot\sqrt{\varepsilon_\parallel^{(II)}(\omega)}\cos\theta_1^{(np)}. \tag{2.137}$$

From (2.136) and (2.137) for $A_2^{(отр)}$ we have:

$$A_2^{(отр)} = \frac{4\pi P_y^{2\omega}}{\varepsilon_\perp^{(II)}(2\omega) - \varepsilon_\parallel^{(II)}(\omega)} \times \\ \times \frac{\sqrt{\varepsilon_\perp^{(II)}(2\omega)}\cos\theta_2^{(np)} - \sqrt{\varepsilon_\parallel^{(II)}(\omega)}\cos\theta_1^{(np)}}{\sqrt{\varepsilon_\perp^{(II)}(2\omega)}\cos\theta_2^{(np)} + \sqrt{\varepsilon_\perp^{(I)}(2\omega)}\cos\theta_2^{(отр)}} \tag{2.138}$$

or, multiplying the numerator and denominator by $\sqrt{\varepsilon_{\perp}^{(II)}(2\omega)}\cos\theta_2^{(пр)} + {}$ $+\sqrt{\varepsilon_{\parallel}^{(II)}(\omega)}\cos\theta_1^{(пр)}$ and noting that according to (2.133)

$$\varepsilon_{\perp}^{(II)}(2\omega)\cos^2\theta_2^{(пр)} - \varepsilon_{\parallel}^{(II)}(\omega)\cos^2\theta_1^{(пр)} = \varepsilon_{\perp}^{(II)}(2\omega) - \varepsilon_{\parallel}^{(II)}(\omega), \tag{2.139}$$

we arrive at formula:

$$A_2^{(отр)} = \frac{4\pi P_y^{2\omega}}{\left[\sqrt{\varepsilon_{\perp}^{(II)}(2\omega)}\cos\theta_2^{(пр)} + \sqrt{\varepsilon_{\perp}^{(I)}(2\omega)}\cos\theta_2^{(отр)}\right]} \times \frac{1}{\left[\sqrt{\varepsilon_{\perp}^{(II)}(2\omega)}\cos\theta_2^{(пр)} + \sqrt{\varepsilon_{\parallel}^{(II)}(\omega)}\cos\theta_1^{(пр)}\right]}. \tag{2.140}$$

The amplitude of the passing wave can be directly calculated from (2.136) and (2.140).

Let us note, first of all, that if the amplitude of the passing wave, as can be seen from (2.136), depends on the relationship of dielectric constants $\varepsilon_{\perp}^{(II)}(2\omega)$ and $\varepsilon_{\parallel}^{(II)}(\omega)$ (when $\varepsilon_{\perp}^{(II)}(2\omega) = \varepsilon_{\parallel}^{(II)}(\omega)$ the wave of nonlinear polarization has a resonance effect on the medium, and stored effects are possible), the resonance term of the form $[\varepsilon_{\perp}^{(II)}(2\omega) - \varepsilon_{\parallel}^{(II)}(\omega)]^{-1}$, in general, does not enter into expression (2.140).[1]

Using (2.140) and (2.123a), it is possible to introduce the "nonlinear" reflectivity of the quadratic medium $R^{(нл)}$, which characterizes the relationship of energy fluxes of the incident wave of basic radiation and "reflected" wave of the second harmonic.

$$R^{(нл)} = \frac{\sqrt{\varepsilon_{\perp}^{(I)}(2\omega)}\cos\theta_2^{(отр)}\cdot|E_2^{(отр)}|^2}{\sqrt{\varepsilon_{\parallel}^{(I)}(\omega)}\cos\theta_1^{(п)}\cdot|E_1^{(п)}|^2}. \tag{2.141}$$

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[1] The last circumstance can have a definite interest for the creation of nonlinear optical devices using skin effects.

In order of magnitude

$$R^{(\text{нл})} \sim [\chi^{2\omega} A_1^{(n)}]^2 \tag{2.142}$$

For the crystal KDP $\chi_{312} = 3\cdot10^{-9}$ CGSE (see [41]), and with a field of $\sim10^5$ V/cm the relative intensity of the second harmonic, appearing on the border of the nonlinear medium, comprises $\sim10^{-12}$ of the intensity of the basic wave. For the crystal GaAs $\chi_{123} = 2.6\cdot10^{-6}$ CGSE (see [180]) and under those same conditions $R^{(\text{нл})} \approx 10^{-6}$. Expression (2.140) permits analyzing the angular structure of the "reflected" wave of the second harmonic; let us note only that in the investigation with the help of (2.140) the dependence of $A_2^{(\text{отр})}$ on θ_1^n should be considered a great dependence of components of the vector of nonlinear polarization $P^{2\omega}$ on the angle of incidence of the wave of the main frequency (see (2.129)).

For surface nonlinear interactions, in exactly the same way as for the volume interactions, of course, the general energy relations of the type (2.42)-(2.43) are fulfilled; however, for their derivation it is impossible to use the approximation of the assigned field.

Although the example examined in this paragraph is one of the simplest, it visually illustrates the method of resolution of problems on surface nonlinear interactions in the approximation of the assigned field. In the end, calculation of the field of "reflected" waves on combination frequencies in the linear medium is reduced to the problem on radiation of the assigned wave of polarization $P^{2\omega}$, or in a more general case, $\mathbf{P}^{\omega_1\pm\omega_2}$, which propagates in the nonlinear medium. Above we examined the case when the vector of the nonlinear polarization is perpendicular to the incidence plane (only one component $P_y^{2\omega} \equiv P_\perp^{2\omega}$ was considered). For an analysis of the general case, it follows to examine still the problem for which the vector of nonlinear polarization lies in the incidence plane ($P_\perp^{2\omega} = 0$; $P_\parallel^{2\omega} \neq 0$). In this case vectors $\mathbf{e}_2^{(\text{отр})}$ and $\mathbf{e}_2^{(\text{пр})}$ also lie in the incidence plane. Relationships between amplitudes $A_2^{(\text{отр})}$, $A_2^{(\text{пр})}$ and $A_1^{(n)}$ can be set if one were to use boundary conditions of the type (2.121).

§ 6. Space-Time Analogy in the Theory of Nonlinear Systems

The application of methods of approximation of the theory of oscillations of nonlinear systems with concentrated constants to the investigation of wave processes in nonlinear dispersive media permits many problems on nonlinear waves to establish the time problems-analogies in correspondence.

Actually, the truncated equations describing the propagation and interaction n of the unmodulated waves are the system n of ordinary differential first-order equations for complex amplitudes. A system of precisely the same type describes oscillations in a weakly nonlinear oscillatory system with n-degrees of freedom. Let us turn, for example, to the problem on the propagation of an unmodulated wave in a linear dissipative medium, examined in § 2 of this chapter. In accordance with (2.20) the change in the complex amplitude A in space is described by the truncated equation:

$$\frac{dA}{dz} - \delta_z A = 0, \text{ where } \delta_z = \frac{2\varkappa\omega\hat{\varepsilon}\sigma e}{c^2 k \cos \hat{ks} \cdot \cos \hat{sz_0}}. \qquad (2.143)$$

It is easy to be convinced that precisely the same structure is seen in the truncated equation describing the process of the change in time of amplitude of free oscillations in the resonator. Actually, let us examine the equation of the linear oscillatory circuit close to the conservative. The equation of such a circuit has the form

$$\ddot{x} + 2\delta_t \dot{x} + \omega_0^2 x = 0. \qquad (2.144)$$

If $\delta_t \sim \mu$ (μ as previously the small parameter), the solution of (2.144) in the first approximation can be sought in the form $x = A(\mu t) \exp i\, \omega_0 t$. Substituting this solution into (2.144) and rejecting terms $\sim\mu^2$, we arrive at the equation:

$$\frac{dA}{dt} - \delta_t A = 0, \qquad (2.145)$$

coinciding according to the formula with (2.143); the process of the change with time of the amplitude of free oscillations in the resonator close to the conservative occurs in exactly the same way as the change in space of the amplitude of the harmonic wave propagating in a weakly absorbing medium.[1] Comparing (2.143) and (2.145), it is possible to establish the direct connection between variables and parameters in problems – analogs. The independent variable t in the time problem corresponds to coordinate z in space, frequency ω – wave number k, initial conditions in the time problem – boundary value problems in space. The analogy can be widespread on nonuniform problems: forced oscillations in the time problem correspond to effected side fields to forced waves (see formulas (2.92)-(2.86) in space. What has been said pertains equally to nonlinear problems. The boundary value problem on the generation of the second harmonic in a dispersive medium, described by equations (2.39) when $\omega_1 = \omega_2 = \omega$ and $\omega_3 = 2\omega$, can be established in correspondence to the problem on free oscillations in a two-circuit system, the resonators of which are tuned to frequencies ω and 2ω and are connected by the nonlinear reactive element. Let us note that the latter was examined as long ago as in the 1930's by A. A. Whitt and G. S. Gorelik [136].

The list of nonlinear analog-problems can easily be expanded.

What has been said, however, denotes that in the theory of nonlinear oscillations of systems with concentrated constants already there are contained solutions of all problems appearing in the theory of nonlinear waves in dispersive media. First, nonlinear wave problems in general are considerably more diverse than the oscillatory (see below). But this is not the only matter. The most interesting practically, in nonlinear optics, for example, are boundary value problems. Their analogs are, in virtue of that

[1]Let us note that the space-time analogy for linear systems was examined in works of P. Ye. Krasnushkin [210].

mentioned above, problems on transition processes in systems with concentrated constants. However, the independent interest of the latter, as a rule, is small and therefore in the theory of nonlinear oscillations the main attention is given to forced oscillations. In connection with this it follows to indicate the distinction between the energy relationships, which usually occur in the theory of nonlinear waves in dielectrics [see, for example (2.43) and (2.57)] and in the theory of nonlinear oscillations of systems with reactive nonlinearity. In the last case usually we are interested in the distribution of energy with respect to frequencies in conditions of stationary forced oscillations; therefore, here relations of the type (2.57) take place not for increases but for total energies corresponding to different oscillations (see, for example, [104], [107]). Distributed energies of forced oscillations in the steady-state operation does not depend on initial conditions. It is necessary to consider also that stationary waves, the existence of which appears possible with a special form of the selected boundary conditions [see, for example (2.72)], are not of course, analogs of the stationary forced oscillations.

The role of the dispersion characteristic of the medium, the analysis of properties of which permits in a nonlinear spatial problem determining waves essentially participating in the process of nonlinear interaction, in the temporal problem is played by the resonance characteristic of the oscillatory system. Here it is possible to be limited to calculation of oscillations of only those frequencies for which the resistance of the oscillatory system is not too small, i.e., for which the system reveals noticeable resonance properties.

Within the bounds of the indicated space-time analogy it is natural to treat the appearance of growing waves in the nonlinear medium as "instability in space."[1] With this the regions of

[1]In the theory of plasma waves the term "convection instability" is also accepted [135]; "instability with time" is then called "absolute" instability.

instability in space for the system of ordinary differential equations of the type, for example, of equations (2.60) can be found with the help of the usual criteria of stability, developed in detail in the theory of oscillations. Such an approach proves to be very effective, for example, in the theory of parametric amplifiers of the traveling wave (see [133, 134], and also Chapter IV of this book). From this point of view the parametric amplifier of the traveling wave is, obviously, a spatial analog of the parametric generator with concentrated constants. The space-time analogy can appear useful in the search for new nonlinear effects in optics, since it permits setting in conformity the nonlinear waveguide analogy-systems with concentrated constants studied thoroughly in radio physics. In proceeding in this way, it is possible to construct, for example, a wave analog of the phenomenon of the forcing of oscillations, which plays a very important role in radio engineering of systems with concentrated constants. The wave analog of the forcing is, obviously, the change in phase speed of the wave propagating in the medium with dissipative nonlinearity, which is connected with the effect on the medium of the external field. Examples of a similar type can be multiplied: it is possible to construct, in any case, theoretically, wave analogies of such oscillatory phenomena as the mutual synchronization, asynchronous interactions and so forth. At the same time, on the path of their experimental realization considerable difficulties can be encountered. The use of the space-time analogy proves to be very useful not only in examining the dynamic but also the statistical wave problems. For example, problems on the influence of side fluctuating forces on the course of nonlinear interactions in dispersive media have much in common with problems on the influence of fluctuations on nonlinear oscillations in systems with concentrated constants. In particular, in the investigation of equations of type (2.86) and (2.91) for those cases when force $I(t, r)$ is accidental, methods and results of works on fluctuating phenomena in self-oscillation systems with concentrated constants prove to be very useful (see, for example, [137, 138, 56]).

In conclusion one should once again stress that everything which has been said pertained to wave problems on the propagation and interaction of unmodulated waves moving in one direction.

In the general case the nonlinear waveguide problems are much more numerous than the oscillatory; the latter is connected with the fact that in waveguide problems there are two independent variables (t and r), and independent variable r can be both increased and decreased (in interactions both direct and backward waves can take part).

As the simplest example of the problem, which does not have a direct analog in the theory of oscillations of systems with concentrated constants, it is possible to indicate, for example, the problem on the propagation of a modulated wave in slightly absorbing linear medium [see equation (2.30)]. The indicated equation should be solved with the boundary condition, set at $z = 0$ and having the form $\dot{A}(\mu t, 0) = \dot{A}_0(\mu t)$, i.e., with the boundary condition dependent on the small parameter. In the theory of free oscillations of systems with concentrated constants such a situation is impossible.

§ 7. Generalized Truncated Equations and Laws of Conservations for the Nonlinear Medium with Temporal and Spatial Dispersion[1]

Although the truncated equations derived in §§ 2-3 of this chapter permit solving the majority of problems appearing in the electrodynamics of the weakly nonlinear dispersive medium, in certain cases there is interest in their generalization in the case of the medium with spatial dispersion and fields somewhat more than that of the general form. Such generalization is the subject of this section.

We will assume that the medium is spatially uniform. This means that parameters entering into the expression for the vector

[1]Yu. L. Klimontovich wrote § 7.

of polarization of the medium do not depend on the coordinates. Furthermore, just as in §§ 1-6 of Chapter I, we consider that parameters of the medium clearly do not depend on time, i.e., a temporal homogeneity takes place.

Under these conditions the expression for the vector of electrical induction

$$\mathbf{D}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+4\pi\mathbf{P}(\mathbf{r},t)$$

can be written in the form:

$$\begin{gathered}D_i(\mathbf{r},t)=\int_0^\infty d\tau_1\int d\mathbf{r}_1\varepsilon_{ij}(\tau_1\mathbf{r}_1)E_j(t-\tau_1,\mathbf{r}-\mathbf{r}_1)+\\+\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int d\mathbf{r}_1\int d\mathbf{r}_2\chi_{ijk}(\tau_1,\tau_2,\mathbf{r}_1,\mathbf{r}_2)E_j(t-\tau_1,\mathbf{r}-\mathbf{r}_1)\cdot E_k(t-\tau_1-\tau_2,\mathbf{r}-\\-\mathbf{r}_1-\mathbf{r}_2)+\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int_0^\infty d\tau_3\int d\mathbf{r}_1\int d\mathbf{r}_2\int d\mathbf{r}_3\times\\\theta_{ijkl}(\tau_1,\tau_2,\tau_3,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\cdot E_j(t-\tau_1,\mathbf{r}-\mathbf{r}_1)\cdot E_k(t-\tau_1-\tau_2,\mathbf{r}-\mathbf{r}_1-\mathbf{r}_2)\\\times E_l(t-\tau_1-\tau_2-\tau_3,\mathbf{r}-\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_3).\end{gathered}\tag{2.146}$$

From the twice repeated indices summation is assumed.

When $\chi_{ijk}=0$ and $\theta_{ijkl}=0$, hence we will obtain the well-known expression for the vector of induction of the linear medium [149].

Expressions given in the introduction of §§ 5-6 neglecting the spatial dispersion, follow from (2.146) under conditions

$$\begin{gathered}\varepsilon_{ij}(\tau,\mathbf{r})=\varepsilon_{ij}(\tau)\cdot\delta(\mathbf{r});\\\chi_{ijk}(\tau,\tau',\mathbf{r},\mathbf{r}')=\chi_{ijk}(\tau,\tau')\cdot\delta(\mathbf{r})\delta(\mathbf{r}')\end{gathered}\tag{2.147}$$

etc.

There appears the question as to what measure expression (2.146), correct for the spatial uniform medium, can be used for crystals. In fact the crystals are not spatially uniform, since ttice points are equivalent to remaining points of the crystal.

Expressions (2.146) can be used for the crystal only in the case of a weak spatial dispersion. This takes place if the wavelength is much greater than the lattice constant a, i.e., $a/\lambda \ll 1$. For the optical range this condition is well fulfilled. (For more detail on this see, for example, the survey of V. M. Agranovich and V. L. Ginzburg [150]).

In the case of weak spatial dispersion, the tensor $\varepsilon_{ij}(\omega)$ can be presented in the form (see also [149]-[150]):

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon_{ij}(\omega) + i\gamma_{ijk}(\omega)\cdot k_k + \alpha_{ijkl}k_k k_l + \ldots \tag{2.148}$$

Tensor γ_{ijk} is different from zero only in the optically active crystals. Conditions of symmetry of tensors $\varepsilon_{ij}(\omega)$ $\gamma_{ijk}(\omega)$ and $\alpha_{ijkl}(\omega)$ are determined by relations resulting from the condition of symmetry of the full tensor $\varepsilon_{ij}(\omega_1, \mathbf{k})$ $(\varepsilon_{ij}(\omega_1, \mathbf{k}) = \varepsilon_{ji}(\omega_1, -\mathbf{k})$

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad \gamma_{ijk} = -\gamma_{jik}, \quad \alpha_{ijkl} = \alpha_{jikl}.$$

We will assume that the term, containing γ_{ijk} has the same order of magnitude as does the imaginary part of the tensor $\varepsilon_{ij}(\omega)$, i.e., the order μ. Therefore, in it dissipation can be disregarded. Under this condition tensor γ_{ijk} is real.

Tensor $\varepsilon_{ij}(\omega, \mathbf{k})$ can be presented in the form:

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon'_{ij}(\omega, \mathbf{k}) + i\varepsilon''_{ij}(\omega, \mathbf{k}), \tag{2.149}$$

where ε'_{ij} and ε''_{ij} – real and imaginary parts of the tensor ε_{ij}.

For the crystal

$$\varepsilon'_{ij}(\omega, \mathbf{k}) = \varepsilon'_{ij}(\omega) + \alpha_{ijkl} k_k k_l; \quad \varepsilon'_{ij}(\omega, \mathbf{k}) = \varepsilon'_{ji}(\omega, \mathbf{k});$$
$$\varepsilon''_{ij}(\omega, \mathbf{k}) = \varepsilon'_{ij}(\omega) + \gamma_{ijk}(\omega) \cdot k_k \sim \mu. \quad (2.149a)$$

Below we will examine the general formulas correct for any media; therefore, subsequently we will use formula (2.149), not assuming that ε'_{ij} and ε''_{ji} are determined by formulas (2.149a).[1]

Let us note still that due to the smallness of the spatial dispersion in crystals, it can be disregarded, in nonlinear terms of formula (2.146), i.e., consider that tensors χ_{ijk} and θ_{ijkl} depend only on frequency arguments.

Instead of equation (2.2) for E, here we will use directly the system of Maxwell equations:

$$\operatorname{rot} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{I}(\mathbf{r}, t), \quad \operatorname{div} \mathbf{H} = 0; \quad (2.150)$$
$$\operatorname{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div} \mathbf{D} = 4\pi q(\mathbf{r}, t) \quad (2.151)$$

I, q — densities of side currents and charges.

In combination with expression (2.146), the system (2.150, 2.151) is closed.

In accordance with that said in §§ 1-2 of this chapter and introduction, we will assume that the spatially temporal process is characterized by rapid and slow changes of all functions, and we will present fields E and H in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mu t, \mu \mathbf{r}, t, \mathbf{r}); \quad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mu t, \mu \mathbf{r}, t, \mathbf{r}), \quad (2.152)$$

where μ — small parameter, $\mathbf{r}, t$ — rapid variables.

[1]For the contemporary state of linear crystalloptics, taking into account spatial dispersion, see [213], [150], [214], [215].

Let us produce decomposition into the Fourier integral according to variables

$$\mathbf{E}(\mu t, \mu \mathbf{r}, t, \mathbf{r}) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int d\mathbf{k} \mathbf{E}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}) e^{-i(\omega t - \mathbf{k}\mathbf{r})}, \tag{2.153}$$

$$\mathbf{H}(\mu t, \mu \mathbf{r}, t, \mathbf{r}) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int d\mathbf{k} \cdot \mathbf{H}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}) e^{-i(\omega t - \mathbf{k}\mathbf{r})}. \tag{2.154}$$

In these formulas the Fourier-components are themselves slow functions of coordinates and time. Such decompositions are expedient, if spectral functions are different from zero only at sufficiently great values of ω and k, i.e., there should exist ω_{min} and k_{min}, which satisfy for example, conditions

$$\omega_{\min} E(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}) \gg \frac{\partial E}{\partial \mu t}; \quad k_{\min} E \gg \left| \frac{\partial E}{\partial \mu \mathbf{r}} \right|. \tag{2.155}$$

Let us obtain equations for slowly changing functions $\mathbf{E}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$; $\mathbf{H}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$ (truncated equations). For this at first let us find the expression for function $\mathbf{D}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$.

Just as everywhere, in this book we will consider that nonlinear terms in expressions (2.146) have the order μ, and we will present D in the form of two parts

$$\mathbf{D} = \mathbf{D}^{(л)} + \mathbf{D}^{(нл)}, \tag{2.156}$$

which correspond to the linear and nonlinear parts.

Let us write functions $D^{(л)}$ and $D^{(нл)}$ in the form of (2.153). Since the nonlinear terms have the order μ, then in obtaining the expression for $\mathbf{D}^{(нл)}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$ correct to μ we are not able to consider in (2.146) the dependence of field E on slow variables. As a result, from (2.146) we will obtain the following expression (in order not to limit community let us hold arguments k at tensors $\hat{\chi}$ and $\hat{\theta}$)

$$D_i^{(\kappa,1)}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega' d\mathbf{k}' \chi_{ijk}(\omega, \mathbf{k}, \omega', \mathbf{k}') \times$$
$$\times E_j(\omega-\omega', \mathbf{k}-\mathbf{k}')\cdot E_k(\omega', \mathbf{k}') +$$
$$+\frac{1}{(2\pi)^8}\int_{-\infty}^{\infty} d\omega' d\mathbf{k}' d\omega'' d\mathbf{k}'' \theta_{ijkl}(\omega, \mathbf{k}, \omega', \mathbf{k}', \omega'' \mathbf{k}'') \times$$
$$\times E_j(\omega-\omega', \mathbf{k}-\mathbf{k}')\cdot E_k(\omega'-\omega'', \mathbf{k}'-\mathbf{k}'')\cdot E_l(\omega'', \mathbf{k}'') \qquad (2.157)$$

Here

$$\chi_{ijk}(\omega, \mathbf{k}, \omega', \mathbf{k}') =$$
$$= \int_0^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \int d\mathbf{r}_1 d\mathbf{r}_2 \chi_{ijk}(\tau_1, \tau_2, \mathbf{r}_1, \mathbf{r}_2) e^{-i(\omega\tau_1 + \omega'\tau_2 - \mathbf{k}\mathbf{r}_1 - \mathbf{k}\mathbf{r}_2)} \qquad (2.158)$$

and similarly there can be obtained the formula for θ_{ijkl}.

In order to obtain the expression for $\mathbf{D}^{(s)}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$, we will substitute into (2.146) decomposition (2.153) in functions of the form: $E_j(\mu(t-\tau_1), \mu(\mathbf{r}-\mathbf{r}_1), \omega, \mathbf{k})$, expand in series with respect to τ_1 and r_1 and limit ourselves to linear terms. As a result we will obtain the following expression:

$$D_i^{(s)}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}) = \left(\varepsilon_{ij}(\omega, \mathbf{k}) + i\frac{\partial \varepsilon_{ij}}{\partial \omega}\cdot\frac{\partial}{\partial \mu t} -\right.$$
$$\left. - i\frac{\partial \varepsilon_{ij}}{\partial \mathbf{k}}\cdot\frac{\partial}{\partial \mu \mathbf{r}}\right)\cdot E_j(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}). \qquad (2.159)$$

Using expressions (2.157, 2.159), we will find the expression for the Fourier-component of function $\partial \mathbf{D}(\mathbf{r}, t)/\partial t$.

$$\left(\frac{\partial \mathbf{D}}{\partial t}\right)_{\mu t, \mu \mathbf{r}, \omega, \mathbf{k}} = -i\omega(\mathbf{D}^{(s)} + \mathbf{D}^{(\kappa s)}) + \frac{\partial \mathbf{D}^{(s)}}{\partial \mu t}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k}). \qquad (2.160)$$

Thus, in expression (2.160) we are limited only to terms $\sim\mu$ and reject terms of a higher order of smallness with respect to μ. As was noted in § 2 of this chapter, rejected terms in certain cases

can appear essential. The necessary refinements of the method of successive approximations in these cases are analogous to those developed in the theory of nonlinear oscillations (see [53], [54], and also § 2 of this chapter, where these methods were used in the conclusion of truncated equations of the anisotropic medium).

From (2.157, 2.159) in a zero and first approximation with respect to μ we will obtain the following expression:

$$\left(\frac{\partial D_i^{(0)}}{\partial t}\right)_{\mu t,\mu r,\omega,k} = -i\omega\varepsilon'_{ij}(\omega,k)\cdot E_j(\mu t,\mu r,\omega,k); \tag{2.161}$$

$$\left(\frac{\partial D_i^{(1)}}{\partial t}\right)_{\mu t,\mu r,\omega,k} = \left(\frac{\partial}{\partial\omega}(\omega\varepsilon'_{ij})\frac{\partial}{\partial\mu t} - \omega\frac{\partial\varepsilon'_{ij}}{\partial k}\frac{\partial}{\partial\mu r}\right)E_j(\mu t,\mu r,\omega,k) +$$
$$+\omega\varepsilon''_{ij}(\omega,k)E_j(\mu t,\mu r,\omega,k) - i\omega D_i^{(\text{нл})}(\mu t,\mu r,\omega,k). \tag{2.162}$$

Let us note that expression (2.157) for $\mathbf{D}^{(\text{нл})}$ by changing the variables and introduction with the help of the delta functions of additional integrals can be written in a more convenient form:

$$D_i^{(\text{нл})}(\omega,\mathbf{k}) = \frac{1}{(2\pi)^8}\int d\omega_1 d\omega_2 d\mathbf{k}_1 d\mathbf{k}_2 \times \delta(\omega-\omega_1-\omega_2)\cdot\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\times$$
$$\times\chi_{ijk}(\omega,\mathbf{k},\omega_2,\mathbf{k}_2)\cdot E_j(\omega_1,\mathbf{k}_1)E_k(\omega_2,\mathbf{k}_2) + \tag{2.163}$$
$$+\frac{1}{(2\pi)^{12}}\int d\omega_1 d\omega_2 d\omega_3 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\omega-\omega_1-\omega_2-\omega_3)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\times$$
$$\times\theta_{ijkl}(\omega,\mathbf{k},\omega_2+\omega_3,\mathbf{k}_2+\mathbf{k}_3,\omega_3,\mathbf{k}_3)E_j(\omega_1\mathbf{k}_1)\cdot E_k(\omega_2,\mathbf{k}_2)E_l(\omega_3\mathbf{k}_3).$$

From this formula it follows that the integrand expressions are different from zero only upon fulfillment of these conditions:

$$\omega=\omega_1+\omega_2,\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \tag{2.164}$$

in quadratic and

$$\omega=\omega_1+\omega_2+\omega_3,\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 \tag{2.165}$$

in cubic terms. In quantum language these conditions express laws of the conservation of energy and impulses of interacting quanta.

Now it is already easy to obtain equations for the slowly changing Fourier-component of intensities E and H.

In the zero approximation with respect to μ from equations (2.150, 2.151), considering (2.161), we find

$$[\mathbf{kH}]_i = -\frac{\omega}{c} \varepsilon'_{ij}(\omega, \mathbf{k}) E_j, \quad \mathbf{kH} = 0; \tag{2.166}$$

$$[\mathbf{kE}]_i = \frac{\omega}{c} H_i, \quad k_i \varepsilon'_{ij} E_j = 0. \tag{2.167}$$

Equations of the first approximation can be written in the following form

$$c\,(\mathrm{rot}_{\mu r}\mathbf{H})_i = \frac{\partial}{\partial\omega}(\omega\varepsilon'_{ij})\frac{\partial E_j}{\partial\mu t} - \omega\frac{\partial\varepsilon_{ij}}{\partial\mathbf{k}}\frac{\partial E_j}{\partial\mathbf{r}} - i\omega D_i^{(nl)} + \omega\varepsilon''_{ij}E_j + 4\pi I \tag{2.168}$$

$$c\,\mathrm{rot}_{\mu r}\mathbf{E} = -\frac{\partial\mathbf{H}}{\partial\mu t}; \tag{2.168a}$$

$$\left(\frac{\partial D_i}{\partial\mathbf{r}}\right)_{\mu t, \mu \mathbf{r}, \omega, \mathbf{k}} = 4\pi q(\mu t, \mu\mathbf{r}, \omega, \mathbf{k}); \tag{2.169}$$

$$\frac{\partial}{\partial\mu\mathbf{r}}\mathbf{H} = 0. \tag{2.170}$$

Here it is assumed that quantities I and q are of the order μ. Other possibilities are analyzed in Section 4.2 of this chapter. Equations of the first approximation permit obtaining the dispersion equation and establishing the connection between vectors $\mathbf{E}(\mu t, \mu\mathbf{r}, \omega, \mathbf{k})$ and $\mathbf{H}(\mu t, \mu\mathbf{r}, \omega, \mathbf{k})$.

From the first two equations (2.166, 2.167) we find:

$$[\mathbf{k}[\mathbf{kE}]]_i + \frac{\omega^2}{c^2}\varepsilon_{ij}E_j = 0. \tag{2.71}$$

If, just as above, we use the unit vector e along intensity E, then from (2.71) there follows the equation

$$[\mathbf{ke}]^2 + \frac{\omega^2}{c^2} e_i \varepsilon_{ij} e_j = 0, \tag{2.172}$$

which connects the frequency ω and vectors k and e.

For the isotropic medium tensor ε_{ij} is determined only by one vector k and therefore can be represented in the form (see, for example, [149]):

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \varepsilon_\perp(\omega, \mathbf{k}) + \frac{k_i k_j}{k^2} \varepsilon_\parallel(\omega, \mathbf{k}), \tag{2.173}$$

where $\varepsilon_\perp$ and $\varepsilon_\parallel$ – transverse and longitudinal dielectric constants. For transverse waves $\mathbf{e} \perp \mathbf{k}$, $e_i \varepsilon_{ij} e_j = \varepsilon_\perp$ and from (2.172) the well-known dispersion equation follows:

$$\omega^2 \varepsilon_\perp(\omega, \mathbf{k}) - c^2 k^2 = 0. \tag{2.174}$$

For longitudinal waves $\mathbf{e} \parallel \mathbf{k}$, and from (2.172) we find:

$$\varepsilon_\parallel(\omega, \mathbf{k}) = 0 \tag{2.175}$$

Let us examine in more detail equations of the first approximation. First of all, we will write the law of the conservation of energy.[1] For this we will multiply equation (2.168) by E_i^*, and the complex conjugate equation (2.169) by H* and subtract the second from the first. Using equations (2.170) and the well-known vector identity

$$\operatorname{div} [\mathbf{AB}] = \mathbf{B} \operatorname{rot} \mathbf{A} - \mathbf{A} \operatorname{rot} \mathbf{B},$$

we will obtain equation (we omit parameter μ):

[1]See [38], [194], [218], [219], [233] on laws of conservation in a linear anisotropic dispersive medium.

$$\frac{1}{8\pi}\cdot\frac{\partial}{\partial t}\left(\mathbf{H}^2+E_i\frac{\partial}{\partial\omega}(\omega\varepsilon'_{ij})E_j\right)=-\frac{c}{4\pi}\operatorname{div}Re\,[\mathbf{E}\mathbf{H}^*]+$$
$$+\frac{\omega}{8\pi}\frac{\partial}{\partial r}E_i\frac{\partial\varepsilon'_{ij}}{\partial\mathbf{k}}E_j-\frac{\omega}{4\pi}\operatorname{Im}(\mathbf{D}^{(\kappa\lambda)}\cdot\mathbf{E}^*)-$$
$$-\frac{\omega}{4\pi}E_i\varepsilon''_{ij}E_j-Re\,\mathbf{I}\mathbf{E}^*. \tag{2.176}$$

Let us introduce the designation for magnitude of the electromagnetic energy of the anisotropic dispersive medium, arrived by given ω and $\mathbf{k}$.

$$U(\mu t,\mu\mathbf{r},\omega,\mathbf{k})=\frac{1}{8\pi}\left(\mathbf{H}^2+E_i\frac{\partial\omega\varepsilon_{ij}}{\partial\omega}E_j\right). \tag{2.177}$$

The first two terms of the right side of equation (2.176) determine the energy flow taking into account spatial dispersion; the third one determines the change in energy due to the nonlinear interaction; the fourth – thermal losses and the fifth – work of the side current.

Using equations of zero approximation (2.166, 2.167), we will obtain:

$$\mathbf{H}=\frac{c}{\omega}[\mathbf{k}\mathbf{E}];\quad [\mathbf{E}\mathbf{H}^*]=\frac{c}{\omega}[\mathbf{E}[\mathbf{k}\mathbf{E}^*]]. \tag{2.178}$$

These relations permit excluding the magnetic field strength from equation (2.176) and to writing expressions for energy and energy flow in the form:

$$U=\left(\frac{c^2}{\omega^2}[\mathbf{k}\mathbf{e}]^2+\frac{\partial}{\partial\omega}[\omega e_i\varepsilon'_{ij}e_j]\right)\frac{|E|^2}{8\pi}=\frac{1}{\omega}\frac{\partial}{\partial\omega}\omega^2e_i\varepsilon'_{ij}e_j\frac{|E|^2}{8\pi} \tag{2.179}$$

$$\mathbf{S}(\mu t,\mu\mathbf{r},\omega,\mathbf{k})=\left[\frac{2c^2}{\omega}(\mathbf{k}-\mathbf{e}(\mathbf{k}\mathbf{e}))-\omega e_i\frac{\partial\varepsilon'_{ij}e_j}{\partial\mathbf{k}}\right]\frac{|E|^2}{8\pi}. \tag{2.180}$$

To obtain the last expression in (2.179) equation (2.172) is used.

From these formulas let us find the expression for group speed in the anisotropic medium, which is determined by the relation

$$\mathbf{S} = \mathbf{v}_{\text{гр}} U \equiv \frac{\partial \omega}{\partial \mathbf{k}} U; \tag{2.181}$$

$$\mathbf{v}_{\text{гр}} = \frac{2c^2(\mathbf{k} - \mathbf{e}(\mathbf{k}\mathbf{e})) - \omega^2 \frac{\partial}{\partial \mathbf{k}} \left(e_i \varepsilon'_{ij} e_j \right)}{\frac{\partial}{\partial \omega} \left(\omega^2 e_i \varepsilon_{ij} e_j \right)}. \tag{2.182}$$

Using the last formula, it is possible to write the evident expression for the beam vector s defined by formula (2.9).

Let us note that expression (2.182) for $v_{\text{гр}}$ can be obtained directly from equation (2.172). For this it is necessary to differentiate this equation with respect to k and to solve the equation obtained by such means with respect to $\partial\omega/\partial k$. Here we obtain direct proof of the parallelism of vectors of energy flow S and vector of group speed $\partial\omega/\partial k$ (compare with the proof in [38] Russian page 402).

The beam vector s can be standardize, for example, in such a manner so that the following condition is fulfilled:

$$\mathbf{Sn} = 1, \text{ where } \mathbf{n} = \frac{c\mathbf{k}}{\omega} \tag{2.183}$$

(compare with formula (2.9). In the absence of spatial dispersion the thus standardized beam vector can be recorded in the form

$$\mathbf{S} = \frac{\mathbf{k} - \mathbf{e}(\mathbf{ke})}{\mathbf{kn} - (\mathbf{ne})(\mathbf{ke})} = \frac{[\mathbf{e}[\mathbf{ke}]]}{[\mathbf{ne}][\mathbf{ke}]}. \tag{2.184}$$

The penultimate term member of the right side of equation (2.176), which determines damping, can be presented in the form

$$-2\gamma(\omega, \mathbf{k}) \cdot U, \tag{2.185}$$

where $\gamma(\omega, \mathbf{k})$ — damping decrement in the anisotropic dispersive medium. It is determined by expression

$$\gamma(\omega, \mathbf{k}) = \frac{\omega e_i \varepsilon''_{ij} e_j}{\frac{c^2}{\omega^2}[\mathbf{ke}]^2 + \frac{\partial}{\partial\omega}[e_i e_{ij} \omega e_j]} = \frac{\omega^2 e_i \varepsilon''_{ij} e_j}{\frac{\partial}{\partial\omega}[\omega^2 e_i \varepsilon_{ij} e_j]}. \tag{2.186}$$

Let us examine certain special cases of formulas (2.182, 2.186) for the group speed and damping decrement.

In the case of the isotropic medium, tensor ε_{ij} is determined by expression (2.173). Using this expression, for transverse waves from (2.182, 2.186) we will obtain

$$\mathbf{v}^{\perp}_{\text{гр}} = \frac{2c^2\mathbf{k} - \omega^2 \frac{\partial Re\,\varepsilon_{\perp}(\omega, \mathbf{k})}{\partial\mathbf{k}}}{\frac{\partial}{\partial\omega}(\omega^2 Re\,\varepsilon_{\perp}(\omega, \mathbf{k})}; \quad \gamma_{\perp} = \frac{\omega Im\,\varepsilon_{\perp}(\omega, \mathbf{k})}{\frac{\partial}{\partial\omega}(\omega^2 Re\,\varepsilon_{\perp}(\omega, \mathbf{k}))}. \tag{2.187}$$

Appropriate expressions for longitudinal waves have the form

$$\mathbf{v}^{\parallel}_{\text{гр}} = -\frac{\frac{\partial}{\partial\mathbf{k}} Re\,\varepsilon_{\parallel}(\omega, \mathbf{k})}{\frac{\partial}{\partial\omega} Re\,\varepsilon_{\parallel}(\omega, \mathbf{k})}; \quad \gamma_{\parallel} = \frac{Im\,\varepsilon_{\parallel}(\omega, \mathbf{k})}{\frac{\partial}{\partial\omega} Re\,\varepsilon_{\parallel}(\omega, \mathbf{k})} \tag{2.188}$$

In deriving these expressions we used the dispersion equation (2.175) for longitudinal waves.

Without spatial dispersion

$$\mathbf{v}_{rp}^{\perp}=\frac{2c^2\mathbf{k}}{\frac{\partial}{\partial\omega}(\omega^2 Re\,\varepsilon^{\perp})},\quad \mathbf{v}_{rp}^{l}=0. \tag{2.189}$$

Finally, in the absence of polarization, when $\varepsilon_{\perp}=1$, from (2.189) we find $\mathbf{v}_{rp}^{\perp}=\frac{c\mathbf{k}}{k}$.

Let us return now to equation (2.176). Using designations (2.179, 2.180, 2.181, 2.186), let us write it in the form

$$\frac{\partial U}{\partial t}+\frac{\partial}{\partial \mathbf{r}}(\mathbf{v}_{rp}U)=$$
$$-2\gamma(\omega,\mathbf{k})\,U-\frac{\omega}{4\pi}\,Im\,(\mathbf{D}^{(un)}\,\mathbf{E}^{*})-Re\,\mathbf{j}\mathbf{E}^{*}. \tag{2.190}$$

Let us examine the general properties of the right term of the right side of this equation, which describes the nonlinear interaction of the waves. For simplicity we will examine the case of the quadratic medium. However, general properties of the nonlinear member established below remain accurate in general.

Vector $\mathbf{D}^{(un)}$ is determined by expression (2.163). In this expression it is possible to use expressions for tensors χ_{ijk}, and θ_{ijkl} neglecting the dissipative members, since according to the condition the actual dissipative terms of the order μ and terms of the order μ^2 will be disregarded. Therefore, tensors χ_{ijk} θ_{ijkl}, which enter into formula (2.163) for $\mathbf{D}^{(un)}$, possess the property:

$$\chi_{ijk}(\omega,\omega',\mathbf{k},\mathbf{k}')=\chi^{*}_{ijk}(\omega,\omega',\mathbf{k},\mathbf{k}')=$$
$$=\chi_{ijk}(-\omega,-\omega',-\mathbf{k},-\mathbf{k}'). \tag{2.191}$$
$$\theta_{ijkl}(\omega,\omega',\omega'',\mathbf{k},\mathbf{k}',\mathbf{k}'')=\theta^{*}_{ijkl}(\omega,\omega',\omega'',\mathbf{k},\mathbf{k}',\mathbf{k}'')=$$
$$=\theta_{ijkl}(-\omega,-\omega',-\omega'',-\mathbf{k},-\mathbf{k}'',-\mathbf{k}'''). \tag{2.192}$$

Let us examine the integral $\int Im(\mathbf{D}^{(н.л)}\cdot\mathbf{E}^*)\,d\omega d\mathbf{k}$. Leaving only the quadratic term from (2.163) we will obtain:

$$\frac{1}{(2\pi)^4}\int Im\,(\mathbf{D}^{(н.л)}\mathbf{E}^*)\,d\omega d\mathbf{k}=\frac{1}{(2\pi)^{12}}Im\int d\omega\,d\omega'\,d\omega''\,d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}''\times$$
$$\times\delta(\omega-\omega'-\omega'')\cdot\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\chi_{ijk}(\omega,\omega'',\mathbf{k},\mathbf{k}'')\times$$
$$\times E_i^*(\omega,\mathbf{k})\cdot E_j(\omega',\mathbf{k}')\cdot E_k(\omega'',\mathbf{k}''). \qquad (2.193)$$

Here and below instead of $\omega_1, \omega_2, \mathbf{k}_1, \mathbf{k}_2$ in (2.163) we will use designations ω', ω'' $\mathbf{k}, \mathbf{k}', \mathbf{k}''$.

Let us replace in (2.193) variables of integration $\omega, \omega', \omega'', \mathbf{k}, \mathbf{k}', \mathbf{k}''$ for $-\omega, -\omega', -\omega'', -\mathbf{k}, -\mathbf{k}', -\mathbf{k}''$. We will use properties (2.191) of the tensor (2.191) and consider that the Fourier-components possess properties:

$$\mathbf{E}^*(\omega,\mathbf{k})=\mathbf{E}(-\omega,-\mathbf{k}). \qquad (2.194)$$

As a result we will obtain that the integral in (2.193) will turn into a complex conjugate. The imaginary part of the complex expression, which possesses property $a+ib=\overline{a-ib}$, is equal to zero. Thus we obtain the first important property of the nonlinear term:

$$Im\int\mathbf{D}^{(н.л)}\mathbf{E}^*\,d\omega\,d\mathbf{k}=0 \qquad (2.195)$$

Below it will be shown that this condition ensures the fulfillment of the law of conservation of the number of quanta in the nondissipative medium.

Let us now prove the second property of the nonlinear member:

$$\int\omega\,\mathbf{D}^{(н.л)}\mathbf{E}^*\,d\omega\,d\mathbf{k}=0, \qquad (2.196)$$

which ensures fulfillment of the law of conservation of electromagnetic energy in the nondissipative medium.

For the quadratic medium from (2.163) we obtain

$$\frac{1}{(2\pi)^4}\int \omega \mathbf{D}^{(\text{н}\text{л})} \mathbf{E}^* d\omega d\mathbf{k} = \frac{1}{(2\pi)^{12}}\int d\omega d\omega' d\omega'' d\mathbf{k} d\mathbf{k}' d\mathbf{k}'' \times$$
$$\times \omega\delta(\omega-\omega'-\omega'')\cdot\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\chi_{ijk}(\omega,\omega'\mathbf{k},\mathbf{k}')\times$$
$$\times E_i^*(\omega,\mathbf{k})\cdot E_j(\omega',\mathbf{k}')E_k(\omega'',\mathbf{k}''). \quad (2.197)$$

By means of replacement $\omega'\to\omega'', \mathbf{k}'\to\mathbf{k}'', j\to k$ we are convinced t at under the integral in (2.197) it is possible to consider

$$\chi_{ijk}(\omega,\mathbf{k},\omega'',\mathbf{k}'') = \chi_{ijk}(\omega,\mathbf{k},\omega',\mathbf{k}'). \quad (2.198)$$

(Let us remember that frequency permutable relationships for tensors $\hat{\tilde{\chi}}$ and $\hat{\chi}$ coincide).

Let us consider the case when calculation of the spatial dispersion is not essential. Here it is possible to use the result of § 1-6.

From formula (1.131) it follows that tensor $\chi_{ijk}(\tau,\tau')$ does not change with replacement of $i\to k$, $\tau\to\tau'$, i.e.

$$\chi_{ijk}(\tau,\tau') = \chi_{kji}(\tau',\tau) \quad (2.199)$$

Substituting this formula into the second expression (1.84), we will obtain the useful equality:

$$\chi_{ijk}(\omega,\omega') = \chi_{kji}(\omega',\omega). \quad (2.200)$$

Let us make in (2.197) at first the replacement $\omega\to-\omega''$, $\mathbf{k}\to-\mathbf{k}''$, and then $i\to k$. Using formulas (2.194, 2.200, 2.191), we will obtain:

$$\omega\chi_{ijk}(\omega,\omega'')E_i^*(\omega,\mathbf{k})\cdot E_j(\omega',\mathbf{k}')\cdot E_k(\omega'',\mathbf{k}'') =$$
$$= -\omega''\chi_{ijk}(\omega'',\omega)E_i(\omega'',\mathbf{k}'')\cdot E_j(\omega',\mathbf{k}')\cdot E_k^*(\omega,\mathbf{k}) =$$
$$= -\omega''\chi_{ijk}(\omega,\omega'')E_i^*(\omega,\mathbf{k})E_j(\omega',\mathbf{k}')\cdot E_k(\omega'',\mathbf{k}''). \quad (2.201)$$

Producing now replacement $\omega\to-\omega'$, $\mathbf{k}\to-\mathbf{k}'$, and then $i\to j$, we will obtain by the same means

$$\begin{aligned}&\omega\chi_{ikj}(\omega,\omega')E_i^*(\omega,\mathbf{k})E_j(\omega',\mathbf{k}')\cdot E_k(\omega'',\mathbf{k}'')=\\&=-\omega'\chi_{ikj}(\omega';\omega)E_i(\omega',\mathbf{k}')E_j^*(\omega^1,\mathbf{k})E_k(\omega'',\mathbf{k}'')=\\&=-\omega'\chi_{ikj}(\omega,\omega')E_i^*(\omega,\mathbf{k})E_j(\omega',\mathbf{k}')E_k(\omega'',\mathbf{k}'').\end{aligned}\qquad(2.202)$$

Using formulas (2.201, 2.202) and equality (2.198), we can record expression (2.197) in symmetrized form:

$$\begin{aligned}&\frac{1}{(2\pi)^4}\int\omega\mathbf{D}^{(\text{HA})}\mathbf{E}^*d\omega d\mathbf{k}=\frac{1}{3(2\pi)^{12}}\int d\omega\, d\omega'\, d\omega''\, d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{k}''\times\\&\times(\omega-\omega'-\omega'')\delta(\omega-\omega'-\omega'')\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\chi_{ijk}(\omega,\omega'')\times\\&\times E_i^*(\omega,\mathbf{k})\cdot E_j(\omega',k')\cdot E_k(\omega'',\mathbf{k}'').\end{aligned}\qquad(2.203)$$

This expression is equal to zero, since

$$(\omega-\omega'-\omega'')\delta(\omega-\omega'-\omega'')=0.$$

In a similar way we can prove the correctness of equality (2.196) and for the cubic nonlinearity. In order to generalize these results on the case when calculation of spatial dispersion is important, it is necessary to prove the accuracy of equality:

$$\chi_{ijk}(\omega,\omega',\mathbf{k},\mathbf{k}')=\chi_{kji}(\omega',\omega,\mathbf{k}',\mathbf{k}),\qquad(2.204)$$

which is a generalization of equality (2.200).

Condition (2.196) expresses the law of conservation of energy with nonlinear interactions. Let us stress that this equality is valid when in nonlinear terms it is possible to disregard the dissipation.

Under those same conditions it is possible to prove the accuracy of equality:

$$\int \mathbf{k}\,(\mathbf{D}^{(\text{н.л})}\mathbf{E}^{*})\,d\omega\,d\mathbf{k}=0, \qquad (2.205)$$

which expresses the law of conservation of momentum with nonlinear interactions.

Let us prove this equality for the quadratic medium. For this case, using (2.163), we will obtain:

$$\int \mathbf{k}\,(\mathbf{D}^{(\text{н.л})}\mathbf{E}^{*})\,d\omega\,d\mathbf{k}=\frac{1}{3(2\pi)^{8}}\int d\omega\,d\omega'\,d\omega''\,d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}''\times$$
$$\times(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,\delta(\omega-\omega'-\omega'')\,\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,\chi_{ijk}(\omega,\omega'')\times$$
$$\times E_i^{*}(\omega,\mathbf{k})\,E_j(\omega',\mathbf{k}')\,E_k(\omega'',\mathbf{k}''). \qquad (2.206)$$

Using properties (2.200) of tensor χ_{ijk}, we can record for the medium without spatial dispersion of equalities analogous to (2.201, 2.202). This permits presenting expression (2.206) in symmetrized form:

$$\int \mathbf{k}\,(\mathbf{D}^{(\text{н.л})}\cdot E^{*})\,d\omega\,d\mathbf{k}=\frac{1}{(2\pi)^{8}}\int d\omega\,d\omega'\,d\omega''\,d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}''\times$$
$$\times\delta(\omega-\omega'-\omega'')\cdot\mathbf{k}\cdot\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,\chi_{ijk}(\omega,\mathbf{k},\omega'',\mathbf{k}'').$$
$$E_i^{*}(\omega,\mathbf{k})\cdot E_j(\omega',\mathbf{k}')\,E_k(\omega'',\mathbf{k}''). \qquad (2.207)$$

Hence, considering that

$$(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\cdot\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')=0,$$

we arrive at the equality (2.205), i.e., to the law of the conservation of momentum with nonlinear interactions. For generalization in the case of the dispersive medium, it is necessary to prove equality (2.204).

Let us return to equation (2.190). Let us introduce the function

$$N(\mu t,\mu\mathbf{r},\omega\mathbf{k})=\frac{U(\mu t,\mu\mathbf{r},\omega,\mathbf{k})}{(2\pi)^{4}\hbar\omega}, \qquad (2.208)$$

which is the generalization on a nonstationary and nonuniform case of the function, which determines the average number of quanta with given ω, $\mathbf{k}$.

We will multiply equation (2.190) consecutively by ω, 1, $\mathbf{k}/\omega$ and integrate with respect to ω, $\mathbf{k}$. Using the designation (2.208) and using properties (2.195, 2.196, 2.205) of the nonlinear term in equation (2.190), we will obtain three equations:

$$\frac{\partial}{\partial t}\int N d\omega d\mathbf{k} + \frac{\partial}{\partial \mathbf{r}}\int \mathbf{v}_{rp} N d\omega\, d\mathbf{k} = -2\int \gamma(\omega, \mathbf{k})\, N d\omega\, d\mathbf{k} - \frac{1}{(2\pi)^4}\int \frac{Re\, \mathbf{IE}^*}{\hbar\omega} d\omega d\mathbf{k}. \tag{2.209}$$

$$\frac{\partial}{\partial t}\int \hbar\omega\, N\, d\omega\, d\mathbf{k} + \frac{\partial}{\partial \mathbf{r}}\int \mathbf{v}_{rp}\hbar\omega\, N d\omega d\mathbf{k} = -2\int \gamma(\omega, \mathbf{k})\, \hbar\omega N d\omega\, d\mathbf{k} - \frac{1}{(2\pi)^4}\int Re\, \mathbf{IE}^*\, d\omega\, d\mathbf{k}. \tag{2.210}$$

$$\frac{\partial}{\partial t}\int \hbar\mathbf{k} N d\omega\, d\mathbf{k} + \frac{\partial}{\partial \mathbf{r}}\int \mathbf{v}_{rp}\hbar\mathbf{k} N d\omega\, d\mathbf{k} = -2\int \gamma(\omega, \mathbf{k})\, \hbar\mathbf{k} N d\omega d\mathbf{k} - \frac{1}{(2\pi)^4}\int \frac{\mathbf{k}}{\omega} Re\, \mathbf{IE}^*\, d\omega\, d\mathbf{k}. \tag{2.211}$$

These equations constitute, the equation of balance of the number of quanta, energy of quanta, momentum of quanta respectively.

It is possible to write a fourth equation for the angular momentum

$$\int [\mathbf{r}\hbar\mathbf{k}]\, N d\omega\, d\mathbf{k}$$

If the medium is nondissipative, i.e., $\gamma = 0$ and the external current is equal to zero, then equations (2.209-2.211) express three laws of conservation: numbers of quanta, energy and momentum.

If, furthermore, function N does not depend on coordinates, then from (2.209-2.211) we obtain three laws of conservation:

$$\int N\, d\omega\, d\mathbf{k} = \text{const}; \quad \int \hbar\omega\, N\, d\omega\, d\mathbf{k} = \text{const}; \quad \int \hbar\mathbf{k} N\, d\omega\, d\mathbf{k} = \text{const}. \tag{2.212}$$

If, conversely, function N does not depend on time, and the change in space occurs only in a direction of the unit vector z_0, then from equations (2.209-2.211) there are three laws of conservation of flows of quanta in the direction z_0.

Expressions for the flow of the number of quanta and energy of quanta have the form

$$\int \mathbf{z}_0 \mathbf{v}_{rp} N d\omega\, d\mathbf{k} = \frac{1}{(2\pi)^4} \int \frac{\mathbf{z}_0 \mathbf{S}}{h\omega} d\omega\, d\mathbf{k} = \mathrm{const}; \tag{2.213}$$

$$\int h\omega \mathbf{z}_0 \mathbf{v}_{rp} N\, d\omega\, d\mathbf{k} = \frac{1}{(2\pi)^4} \int \mathbf{z}_0 \mathbf{S}\, d\omega\, d\mathbf{k} = \mathrm{const}. \tag{2.214}$$

Vector S is determined by expression (2.180). Neglecting the spatial dispersion $\mathbf{S} = \frac{c}{4\pi} \mathrm{Re}[\mathbf{E}\mathbf{H}^*]$, expressions (2.213, 2.214) turn into

$$Re \int \frac{\mathbf{z}_0 [\mathbf{E}\mathbf{H}^*]}{\omega} d\omega\, d\mathbf{k} = \mathrm{const}, \quad Re \int \mathbf{z}_0 [\mathbf{E}\mathbf{H}^*]\, d\omega\, d\mathbf{k} = \mathrm{const}. \tag{2.215}$$

Let us examine in more detail the equation (2.211). When $\gamma = I = 0$ we will write it in the form

$$\frac{\partial}{\partial t} \int \hbar k_i N d\omega\, dr = -\frac{\partial}{\partial r} S_{ij} = 0. \tag{2.216}$$

The tensor of stresses S_{ij} is determined by expression

$$S_{ij} = \int k_i \frac{S_j}{\omega} d\omega\, d\mathbf{k} = \int \hbar k_i v_{r,j} N d\omega\, d\mathbf{k}. \tag{2.217}$$

Neglecting the spatial dispersion

$$S_{ij} = \frac{c}{4\pi (2\pi)^4} \int \frac{k_i Re [\mathbf{E}\mathbf{H}^*]_j}{\omega} d\omega\, d\mathbf{k},$$

when $\mathbf{e} \perp \mathbf{k}$.

$$S_{ij} = \frac{c^2}{4\pi (2\pi)^4} \int \frac{k_i k_j}{\omega^2} |E|^2 d\omega\, d\mathbf{k}.$$

Let us clarify the condition under which equation (2.37) can be obtained. For this with the help of equations of aero approximation we will exclude H from equations (2.169, 2.170). As a result we will obtain equations for complex amplitudes $E_l(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$;

$$\frac{1}{\omega}\frac{\partial}{\partial\omega}\omega^2\varepsilon_{lj}\frac{\partial E_j}{dt} - \frac{c^2}{\omega}(\mathrm{rot}\,[\mathbf{kE}] + [\mathbf{k}\,\mathrm{rot}\,\mathbf{E}]) - \omega\frac{\partial\varepsilon_{lj}}{\partial\mathbf{k}}\frac{\partial E_j}{\partial\mathbf{r}} = = -\omega\varepsilon_{lj}^{\cdot}E_j + i\omega D_l^{(\text{n.a})} - 4\pi I_l. \tag{2.218}$$

In this equation we multiply scalarly by E_l^*, take the real part and use designations (2.179-2.182), then we will obtain equation (2.190). Here the vector identity (2.21) is used.

In equations (2.218) $D_l^{(\text{n.a})}$ is determined by expression (2.163). Due to the nonlinearity waves with various ω and k are linked. Therefore, equations (2.218) actually represent an infinite system of integrodifferential equations for amplitudes E_1 with different ω and k or for an infinite number of waves. It is essential, however that under certain conditions the solution of equations (2.218) can be presented in the form of the sum of a small number of waves.

Let us consider the case of quadratic nonlinearity. Let us assume that function $E_l(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$ is different from zero only in three points of the four-dimensional space ω, k, connected by conditions (2.33, 34)

$$\omega_1 + \omega_2 = \omega_3;\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3. \tag{2.219}$$

Function $\mathbf{E}(\mu t, \mu \mathbf{r}, \omega, \mathbf{k})$ can be presented in the form

$$\mathbf{E} = (2\pi)^4 (\mathbf{A}_{(1)}\cdot\delta(\omega-\omega_1)\cdot\delta(\mathbf{k}-\mathbf{k}_1) + \mathbf{A}_{(2)}\cdot\delta(\omega-\omega_2)\cdot\delta(\mathbf{k}-\mathbf{k}_2) + + \mathbf{A}_{(3)}\cdot\delta(\omega-\omega_3)\cdot\delta(\mathbf{k}-\mathbf{k}_3)) \tag{2.220}$$

Let us substitute this expression into equation (2.218). We will assume that value ω, k in (2.218) is close to ω_1, k_1. Here

values $\omega_1, \omega_2, \omega_3, \mathbf{k}_1\ \mathbf{k}_2, \mathbf{k}_3$ are such that

$$\omega_\alpha - \omega_\beta \gg \gamma_\alpha,\ \gamma_\beta;\ k_\alpha - k_\beta \gg \frac{\gamma_\alpha k_\alpha}{\omega_\alpha},\ \frac{\gamma_\beta k_\beta}{\omega_\beta}$$
$$\alpha, \beta = 1, 2, 3.\ \alpha \neq \beta. \quad (2.221)$$

Let us integrate with respect to ω, $\mathbf{k}$ over a small region surrounding point ω_3, $\mathbf{k}_3$. As a result we will obtain the following equation:

$$\frac{1}{\omega_3}\frac{\partial \omega_3^2}{\partial \omega_3}\varepsilon'_{lj}\frac{\partial A_{j(3)}}{\partial t} - \frac{c^2}{\omega_3}\left(\mathrm{rot}\left[\mathbf{k}_3 \mathbf{A}_{(3)}\right] + \left[\mathbf{k}_3 \mathrm{rot}\, \mathbf{A}_{(3)}\right]\right) -$$
$$-\omega_3 \frac{\partial \varepsilon'_{lj}}{\partial \mathbf{k}_3}\frac{\partial A_{l(3)}}{\partial \mathbf{r}} = -\omega_3 \varepsilon''_{lj} A_{j(3)} + i\omega_3 \{\chi_{ljk}(\omega_3 \mathbf{k}_3 \omega_2 \mathbf{k}_2) \times$$
$$\times A_{j(1)} A_{k(2)} + \chi_{ljk}(\omega_3, \mathbf{k}_3, \omega_1, \mathbf{k}_1) A_{j(2)} A_{k(1)}\}. \quad (2.222)$$

Here $\mathbf{I}=0$.

Let us introduce the unit vector along $\mathbf{A}_{(l)}(\mathbf{A}_{(l)}=\mathbf{e}_{(l)}A_{(l)})$, multiply equation (2.222) by $e_{l(3)}$, and use property (2.198).

$$\frac{1}{\omega_3}\frac{\partial}{\partial \omega_3}\left(\omega_3^2 e_{(3)l}\varepsilon'_{lj} e_{(3)j}\right)\frac{\partial A_3}{\partial t} + 2\frac{c^2}{\omega_3}\left[\mathbf{e}_3 \left[\mathbf{k}\mathbf{e}_3\right]\right]\frac{\partial A_3}{\partial \mathbf{r}} -$$
$$-\omega_3 \frac{\partial}{\partial k_3}\left(e_{(3)l}\varepsilon_{lj}\, e_{(3)j}\right)\frac{\partial A_3}{\partial t} = -\omega_3 e_{(3)l}\varepsilon''_{lj} e_{(3)j} A_3 +$$
$$+ 2i\omega_3 \chi_{ljk}(\omega_3, \mathbf{k}_3, \omega_2 \mathbf{k}_2)\, e_{(3)l} e_{(1)j} e_{(2)k} A_1 A_2. \quad (2.223)$$

We write this equation neglecting the spatial dispersion. From formulas (2.182) in this case it follows that

$$\frac{1}{\omega}\frac{\partial \omega^2}{\partial \omega} e_l \varepsilon'_{lj} e_j = \frac{2c^2}{\omega}\left[\mathbf{e}\left[\mathbf{k}\mathbf{e}\right]\right]\frac{\mathbf{v}_{\text{гр}}}{v_{\text{гр}}^2} = \frac{2c^2}{\omega}\left[\mathbf{e}\left[\mathbf{k}\mathbf{e}\right]\right]\mathbf{s}.$$

Using this, we will obtain the following equation:

$$\left[\mathbf{e}_3\left[\mathbf{k}\mathbf{e}_3\right]\right]\mathbf{s}\frac{\partial A_3}{\partial t} + \left[\mathbf{e}_3\left[\mathbf{k}\mathbf{e}_3\right]\right]\frac{\partial A_3}{\partial \mathbf{r}} = -\frac{\omega_3^2}{2c^2} e_{(3)l}\varepsilon''_{lj} e_{(3)j} A_3 +$$
$$+ i\frac{\omega_3^2}{c^2}\chi_{ljk}(\omega_3, \omega_2)\, e_{(3)l} e_{(1)j} e_{(2)k} A_1 A_2. \quad (2.224)$$

This equation coincides with the third equation (2.37). In comparing them, it follows to consider that

$$\mathbf{e}_1\hat{\alpha}\mathbf{e}_2 = \frac{\omega_3^2}{2c^2} e_{(3)i} e_{ij}^{*} e_{(3)j};\ \beta = \frac{1}{2c^2} e_{(3)i}\chi_{ijk}(\omega_3\omega_2)\, e_{(1)j} e_{(2)k}.$$

The equation for A_2 is obtained from (2.224) by replacement of $\omega_3 \to -\omega_2$ $\mathbf{k}_3 \to -\mathbf{k}_2, 2 \to 3, i \to k.$ With such replacement, according to (2.200, 2.191) function

$$\chi_{ijk}(\omega_3, \omega_2) \cdot e_{(3)i} \cdot e_{(1)j} \cdot e_{(2)k} \tag{2.225}$$

does not change. Furthermore, $A_l(-\omega_l, -\mathbf{k}_l) = A_l^*(\omega_l, \mathbf{k}_l).$

In order to obtain equation for A_1, it is necessary to use equality (2.198) and produce replacement of $\omega_3, \mathbf{k}_3 \to -\omega_1 - \mathbf{k}_1,\ 3 \to 1,\ i \to j.$ With such replacement function (2.225) again remains constant.

If one were to substitute expression (2.220) in laws of conservation, then we will obtain corresponding expressions in the approximation of three waves. In particular, from (2.214) (2.44) follows.

For cubic nonlinearity instead of (2.220), in general, it is necessary to use a combination of four waves, the frequencies and wave numbers of which satisfy conditions

$$\omega_1 + \omega_2 + \omega_3 = \omega_4;\ \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 \tag{2.226}$$

and (2.221).

It is essential that conditions (2.226) can be satisfied by two waves if

$$\omega_1 = \omega_2 = \omega_3 = \omega;\ \omega_4 = 3\omega;\ \mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k};\ \mathbf{k}_4 = 3\mathbf{k}.$$

It is necessary to remember that functions A_i should satisfy equations of zero approximation (they are amplitudes of natural waves of the linear medium). At the same time, in the presence of dispersion conditions (2.219) and (2.226) for natural waves cannot be fulfilled; small deviations from conditions (2.219) and (2.226) can be considered by means of the introduction of "vectors of detuning" $\Delta_1 \sim \mu$; $\Delta_2 \sim \mu$; $\Delta_3 \sim \mu$ (see also §§ 3-4 of this chapter). Let us note also that the presentation of the field in the form of superposition of harmonic waves (2.220) can be replaced by a more general representation in the form of the sum of wave packets; here δ – function in (2.220) should be replaced by functions of finite width.

As was shown in the introduction, in those cases when the aproximation of the solution by a small number of waves is unsatisfactory, sometimes it is more convenient to use another extreme approximation, which consists in the full disregard of the dispersion. In this case instead of expressions (2.146, 2.148) we have

$$D_i(\mathbf{r},t) = \varepsilon_{ij} E_j(\mathbf{r},t) + \chi_{ijk} E_j(\mathbf{r},t) \cdot E_k(\mathbf{r},t) + \\ + \theta_{ijkl} E_j(\mathbf{r},t) E_k(\mathbf{r},t) E_l(\mathbf{r},t). \qquad (2.227)$$

Here ε_{ij}, γ_{ijk}, θ_{ijkl} – constants of the tensor.

Intensities of fields will again be presented in the form of (2.152). Equations of zero approximation have the form

$$\begin{aligned} \mathrm{rot}_r \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}^{(n)}}{\partial t}, \quad \mathrm{div}_r \mathbf{H} = 0; \\ \mathrm{rot}_r \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \mathrm{div}_r \mathbf{D}^{(n)} = 0 \end{aligned} \qquad (2.228)$$

and represent linear equations of the anisotropic medium with the tensor ε_{ij}.

$$\mathrm{rot}_{\mu r}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}^{(л)}}{\partial \mu t} + \frac{1}{c}\frac{\partial \mathbf{D}^{(нл)}}{\partial t} + \frac{4\pi}{c} I; \ \mathrm{div}_{\mu r}\mathbf{H} = 0;$$
$$\mathrm{rot}_{\mu r}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial \mu t}; \ \mathrm{div}_{\mu r}\mathbf{D}^{(л)} + \mathrm{div}_{r}\mathbf{D}^{(нл)} = 4\pi q. \tag{2.229}$$

From equations (2.228) it follows that the dependence on rapid variables enters only into combustions $t-\mathbf{v}_{\phi}\mathbf{r}$ (see the introduction).

From (2.228, 2.229) it is possible to obtain in the corresponding approximation telegraph equations, which were used in work [55] for the description of shock waves in electrical lines.

CHAPTER IV

PARAMETRIC EFFECTS IN OPTICS

§ 1. Introduction

Disturbance of the principle of superposition in a nonlinear medium leads, as was shown in Chapter II, to interactions of waves of different frequencies. An important class of such interactions is the parametric interactions (see § 4 of Chapter II) appearing in the nonlinear dispersive medium, which is excited by an intense electromagnetic wave – so-called pumping.

Parametric wave interactions were repeatedly observed in the range of decimeter and centimeter radio waves (see, for example, experimental works [20, 151-153]). Below we will examine peculiarities of parametric interactions in reference to optics, where in this chapter the main attention will be given to interactions for which the frequency of pumping and frequency of interacting waves ω_i are either comparable, $\omega_H \sim \omega_i$ or $\omega_H > \omega_i$. The result of such interactions, as was shown in § 4 of Chapter II, can be the amplification of parametrically interacting waves; we will analyze parametric interactions with $\omega_H \ll \omega_i$ (modulation of electromagnetic waves in nonlinear media in Chapter V.

In this chapter the field in the nonlinear medium will be written in the form

$$\mathbf{E}(t, \mathbf{r}) = \mathbf{e}_H A_H(\mu \mathbf{r}) \exp i(\omega_H t - \mathbf{k}_H \mathbf{r}) + \sum_{l=1}^{N} \times$$
$$\times \mathbf{e}_l A_l(\mu \mathbf{r}) \exp i(\omega_l t - \mathbf{k}_l \mathbf{r}) = \mathbf{E}_H + \sum_{l=1}^{N} \mathbf{E}_l.$$

The character of the interaction of waves E_l in the medium, excited by an intense wave of pumping E_H, is determined by nonlinear and dispersion properties of the medium and also by the relations. p of amplitudes A_l and A_H. Here one should pay attention to the fact that in the presence of an intense wave of pumping not only waves of the field (through electron or ionic oscillations possessing dipole moments, different from zero, see Chapter I) dipole moments, but also waves of field and completely symmetric oscillations of molecules of the medium not possessing a dipole moment (one-sided in effect of such oscillations on the field of the light wave, "passive" combination scattering was discovered in the 1930's by L. I. Mandel'shtam and G. S. Landsberg and Raman [154-155]), and electromagnetic and acoustic waves etc.

In accordance with what has been said, in this chapter we will examine separately parametric interactions of two types. We will subsequently call "nonresonant" the parametric interactions appearing in the medium, the nonlinear properties of which are described by equations of the form (1.17a) or (1.41a). (Here frequencies of interacting waves can be changed in rather wide limits; the possibility of the appearance of accumulating effects is determined by dispersion properties of the medium). Conversely, parametric effects appearing with the interaction of completely symmetric oscillations of molecules (frequency Ω_0) and oscillations with a dipole moment different from zero have a highly marked resonance character – the frequency Ω_0 is fixed and is the parameter of the medium.

The theory of parametric interactions in which acoustic wave participate is analogous to the theory of "nonresonant" electromagnetic interactions.

§ 2. Nonresonant Parametric Amplification of Traveling Waves in a Quadratic Medium

Within the bounds of the three-frequency interaction, as was shown in § 4 of Chapter II, two parametric effects are possible:

1. Parametric amplification of two weak waves with frequencies

$$\omega_1 + \omega_2 = \omega_н \qquad (4.1)$$

(so-called parametric amplification with high-frequency pumping).

2. Parametric conversion of energy of weak waves with frequencies ω_1 and ω_2, which satisfy the relation:

$$\omega_1 + \omega_н = \omega_2. \qquad (4.2)$$

The number of parametric effects in the quadratic medium can be considerably expanded if its dispersion properties allow consecutive three-frequency interactions. Taking into account, for example, two consecutive three-frequency interactions, one should examine the interaction of waves with frequencies ω_1, ω_2, ω_3 and ω_4, which satisfy relation

$$\omega_1 + \omega_2 = \omega_н;\ \omega_1 + \omega_н = \omega_3;\ \omega_2 + \omega_н = \omega_4 \qquad (4.3)$$

(further we will be convinced that (4.3) is one of the possible variants of parametric amplification with low-frequency pumping).

Let us turn to a more detailed investigation of the enumerated effects.

2.1. Parametric Amplification with High-Frequency Pumping

Stored interactions of the type (4.1) can be realized, obviously, in media allowing coherent generating of the second harmonic (see § 2 of Chapter III). In a uniaxial crystal of the KDP or ADP type, pumping should excite the extraordinary wave, and oscillations of frequencies ω_1 and ω_2 (here they will be called also frequency of the signal $\omega_1 \equiv \omega_c$ and difference frequency $\omega_2 \equiv \omega_p$) should excite

ordinary waves. At the assigned frequencies ω_c, ω_p, and ω_H the fixed direction can be found, in which there is carried out one-dimensional coherent interaction of the type (4.1) (for such interaction $v_c^o = v_p^o - v_H^e$).

Fig. 4-1 shows the method of graphic determination of the direction of synchronism in a uniaxial crystal. For θ_0 (compare (3.8) we have

$$\theta_0 = \arcsin \sqrt{\frac{[(k_c^o + k_p^o)\lambda_H]^{-2} - [n_H^o]^{-2}}{[n_H^e]^{-2} - [n_H^o]^{-2}}}.$$

If the frequency and direction of the wave vector of pumping are fixed, the obtaining of considerable coherent lengths with a frequency shift ω_c can be attained due to the use of two-dimensional interactions. The truncated equations, which describe parametric amplification with high-frequency pumping, were already derived in Chapter II.

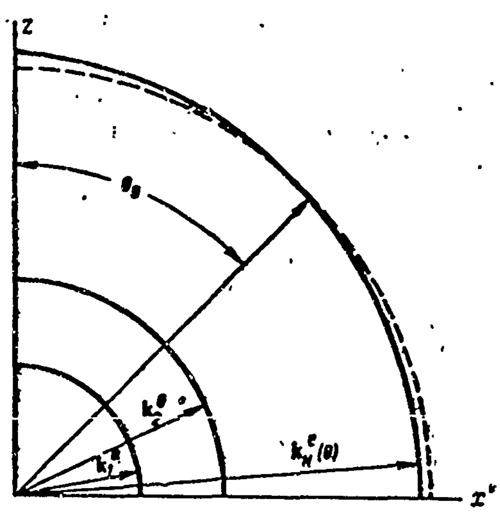

Fig. 4-1. Determination of the direction in which one-dimensional parametric amplification with high-frequency pumping occurs. In the first quadrant of plane z', x' sections of surfaces of wave numbers k_1^o, k_2^o (solid circumferences) and the section k_H^e. The point of intersection of the circumference of the radius $k_1^o + k_2^o$ (dashed line) with curve $k_H^e(\theta)$ determines the direction of the synchronism.

Considering in (2.39) $A_3 = A_H$ = const and passing to the real amplitudes and phases [see (3.19)], we obtain the equation of the form (in contrast to (2.39) we consider here damping in the medium)

$$\frac{dA_1}{dz} + \sigma_1 A_H A_2 \sin\Phi + \delta_1 A_1 = 0; \qquad (4.4a)$$

$$\frac{dA_2}{dz} + \sigma_2 A_H A_1 \sin\Phi + \delta_2 A_2 = 0; \qquad (4.4b)$$

$$\frac{d\Phi}{dz} + \Delta + \left(\sigma_1 \frac{A_2}{A_1} + \sigma_2 \frac{A_1}{A_2}\right) A_H \cos\Phi = 0. \qquad (4.4c)$$

Here $\Phi = \phi_1 + \phi_2$, parameters σ_1, σ_2, δ_1, δ_2 and Δ have the same meaning as analogous parameters introduced in Chapter III [see formulas (3.23)-(3.24); (3.44)].

Equations (4.4) must be solved with boundary conditions, set when $z = 0$.

$$A_1(0) = A_{10};\; A_2(0) = A_{20};\; \Phi(0) = \Phi_0. \qquad (4.4d)$$

As was shown in Chapter II, for the special case $\Delta = \delta_1 = \delta_2 = 0$ equations (4.4) allow the existence of waves by growing exponentially in space. System (4.4) can be unstable in space. In order to determine regions of instability in general $\Delta \neq 0$; $\delta_1 \neq 0$ and $\delta_2 \neq 0$ we will use the usual procedure of investigation on the stability of systems of the third order developed in the theory of oscillations. Introducing small variations of amplitudes $A_{1,2}$ and phases Φ

$$A_1 = A_{10} + \alpha_1;\; A_2 = A_{20} + \alpha_2;\; \Phi = \Phi_0 + \psi, \qquad (4.5)$$

substituting (4.5) into equation (4.4), expanding the right sides of equations in Taylor series with respect to small α_1, α_2, ψ and being limited to terms of the first order of smallness, a system of three differential first-order equations for α_1, α_2 and ψ. Calculating coefficients of the characteristic equation of this system and using

the criterion Routh-Hurwitz [54], we will obtain the conditions of parametric amplification of traveling waves in a quadratic medium with high-frequency pumping:

$$|\Delta| < (\delta_1 + \delta_2)\sqrt{\frac{\sigma_1 \sigma_2 A_H^2}{\delta_1 \delta_2} - 1}. \tag{4.6}$$

The graphic image of regions of amplification (4.6) is given on Fig. 4-2[1]. From (4.6) it is clear that parametric amplification in general is impossible with an amplitude of pumping smaller than the threshold value:

$$A_{\text{пор}}\big|_{\Delta=0} = \sqrt{\frac{\delta_1 \delta_2}{\sigma_1 \sigma_2}} = \sqrt{\frac{(e_1 \hat{\sigma} e_1)(e_2 \hat{\sigma} e_2)}{(e_1 p^{\omega_H - \omega_2})(e_2 p^{\omega_H - \omega_1})\,\omega_1 \omega_2}}. \tag{4.7}$$

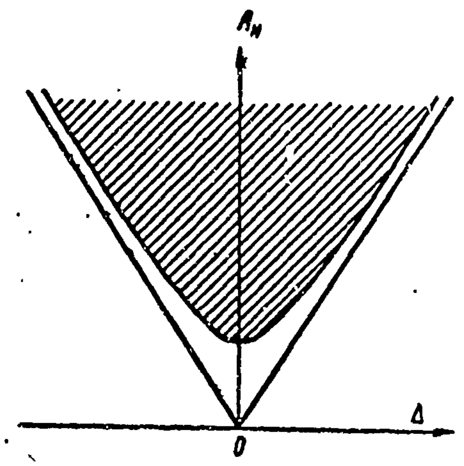

Fig. 4-2. Regions of parametric amplification of traveling waves in a quadratic medium with high-frequency pumping (region of "instability in space"): shaded is the region of instability when $\delta_{1,2} \neq 0$.

With a growth in $|\Delta|$ the threshold value of the amplitude of pumping increases. The obtaining of analytic formulas for the amplitude and phases of growing waves, in general, when $\Delta \neq 0$, $\delta_1 \neq 0$ and $\delta_2 \neq 0$, prove to be very laborious.

[1]Let us note that condition (4.6) has the same form as that condition of instability of a parametrically excited oscillatory circuit known in the theory of oscillations (see [147]-[148]).

For $\Delta = \delta_1 = \delta_2 = 0$ the solution of system (4.4) has the form (see also (2.62))

$$\sin\Phi = \pm 1; \tag{4.8a}$$

$$A_1(z) = a_1 e^{\Gamma_0 z} + b_1 e^{-\Gamma_0 z}; \tag{4.8b}$$

$$A_2(z) = a_2 e^{\Gamma_0 z} + b_2 e^{-\Gamma_0 z}, \tag{4.8c}$$

$$\Gamma_0 = \sqrt{\frac{(2\pi)^2 \omega_1^2 \omega_2^2 A_H^2 \left(\mathbf{e}_1 \mathbf{p}^{\omega_H - \omega_2}\right)\left(\mathbf{e}_2 \mathbf{p}^{\omega_H - \omega_1}\right)}{c^4 k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \cdot k_2 \cdot \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \widehat{\mathbf{s}_2 \mathbf{z}_0}}}. \tag{4.8d}$$

Constants a_1, b_1, a_2 and b_2 can be determined with the help of boundary conditions. The most typical in the problem on parametric amplification are boundary conditions where only one of the amplitudes A_{10} and A_{20} is different from zero; in the medium with variable parameters the signal is introduced from without, and the wave necessary for amplification of the difference frequency appears already inside the medium. When $A_{10} \neq 0$; $A_{20} = 0$ from (4.8) we have:

$$A_1(z) = A_{10} \cdot \operatorname{ch} \Gamma_0 z; \tag{4.9a}$$

$$A_2(z) = A_{10} \sqrt{\frac{\omega_2^2 k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}}{\omega_1^2 k_2 \cdot \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cdot \cos \widehat{\mathbf{s}_2 \mathbf{z}_0}}} \operatorname{sh} \Gamma_0 z \tag{4.9b}$$

(see Fig. 4-3).

From (4.8) it follows that the factor of accretion depends on the frequency of the signal as $\Gamma_0 \sim \omega_c(\omega_H - \omega_c)$. The last expression reaches a maximum when $\omega_c = \frac{\omega_H}{2}$, and such conditions of the parametric amplifier can be called degenerated. With the departure of ω_c from $\frac{\omega_H}{2}$, Γ_0 decreases, and when $\omega_c = 0; \omega_H$ the parametric amplification in the examined approximation in general, vanishes (let us recall that the above-cited analysis is carried out with the help of truncated equations of the first approximation). It is obvious, in reality, that the frequency range in which there is parametric amplification proves to be considerably smaller and is determined, in the first place, by linear dispersion characteristics of the medium. (Formula

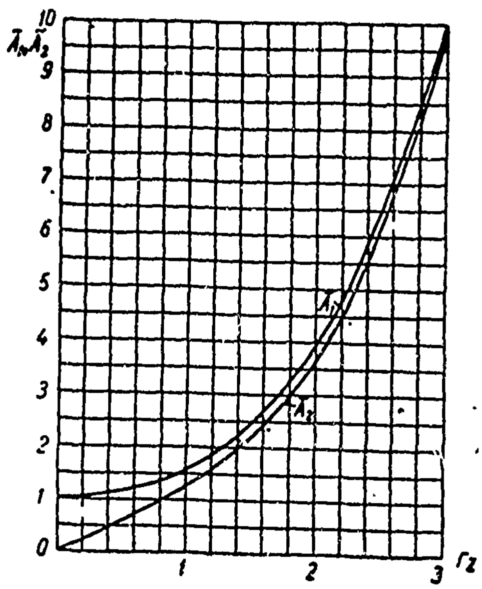

Fig. 4-3. Graphs of the change in space of amplitudes of the signal A_1 (z) and difference frequency A_2 (z) for boundary conditions $A_{10} \neq 0$; $A_{20} = 0$. Plotted along the axis of the ordinates are given amplitudes $\tilde{A}_1 = \frac{A_1}{A_{10}}$; $\tilde{A}_2 = \frac{A_2}{A_{10}} \sqrt{\frac{\omega_1^2 k_2 \cos \hat{k_1 s_2} \cos \hat{s_2 z_0}}{\omega_2^2 k_1 \cos \hat{k_1 s_1} \cos \hat{s_1 z_0}}}$. Along the axis of the abscissas $\tilde{z} = \Gamma z$

(4.8), (4.9) pertain to the case $\Delta = 0$).

The solution to equations (4.4) is when $\Delta = 0$, but losses different from zero can be obtained with the help of a change in variables (3.52) useful in the case $\delta_1 = \delta_2 = \delta$.

For the case $\delta z \ll 1$ most interesting in practice, calculation of losses is reduced to replacement of the parameter of accretion Γ_0 by

$$\Gamma_\delta = \Gamma_0 - \delta. \tag{4.10}$$

When $\Delta \neq 0$ the effectiveness of the parametric interaction is determined by parameter $\tilde{\Delta} = \frac{\Delta}{2\sigma_1 A_H}$ (compare Chapter III); when $\tilde{\Delta} > 1$ interaction practically vanishes.

2.2. Parametric Conversion of Frequency

The interaction of (4.2) is described by equations (2.66)-(2.67), derived in Chapter II. As was already indicated in Chapter II, these equations do not have exponentially growing solutions. The interaction of waves of frequencies ω_1 and $\omega_2 = \omega_H + \omega_1$ has a character of spatial beats. In order to calculate the form of the spatial beats, we will use the boundary conditions. Let us assume that

$$A_{10} \neq 0;\ A_{20} = 0. \tag{4.11}$$

According to (2.69) we have: (here and further, instead of designations of equations (2.66) and (2.67), we use designations accepted in this chapter)

$$A_1(z) = a_1 e^{\Gamma_0 z} + b_1 e^{-i\Gamma_0 z}; \tag{4.12}$$

$$A_2(z) = -\frac{c^2 k_1 \cos \hat{k_1 s_1} \cos \hat{k_1 z_0}}{2\pi\omega_1^2 (e_1 p^{\omega_H - \omega_2})} \Gamma_0 \left(a_1 e^{i\Gamma_0 z} - b_1 e^{-i\Gamma_0 z}\right). \tag{4.13}$$

With boundary conditions (4.11), from (4.13) we have $a_1 = b_1$ and, consequently,

$$A_1(z) = A_{10} \cos \Gamma_0 z; \tag{4.14a}$$

$$A_2(z) = A_{10} \sqrt{\frac{\omega_2^2 k_1 \cos \hat{k_1 s_1} \cos \hat{s_1 z_0}}{\omega_1^2 k_2 \cos \hat{k_2 s_2} \cos \hat{s_2 z_0}}} \cdot \sin \Gamma_0 z. \tag{4.14b}$$

Considering that A_i are amplitudes of the electrical field and using (2.42), for energy flows along the z axis we have

$$S_{1z} = [E_1 H_1] z_0 = \left(A_{10}^2 \frac{c}{\omega_1} k_1 \cos \hat{k_1 s_1} \cos \hat{s_1 z_0}\right) \cos \Gamma_0 z; \tag{4.15a}$$

$$S_{2z} = [E_2 H_2] z_0 = \left(A_{10}^2 \frac{c\omega_2}{\omega_1^2} k_1 \cos \hat{k_1 s_1} \cos \hat{s_1 z_0}\right) \sin \Gamma_0 z. \tag{4.15b}$$

Whence

$$\frac{S_{2z\,\text{макс}}}{S_{1z\,\text{макс}}} = \frac{\omega_2}{\omega_1}. \qquad (4.16)$$

Thus, for the interaction (4.2) the energy periodically passes from the wave of frequency ω_1 to the wave of frequency ω_2 and back (see Fig. 4-4). Here the total energy flow

$$[E_1H_1^*]z_0 + [E_2H_2^*]z_0 \neq \text{const}.$$

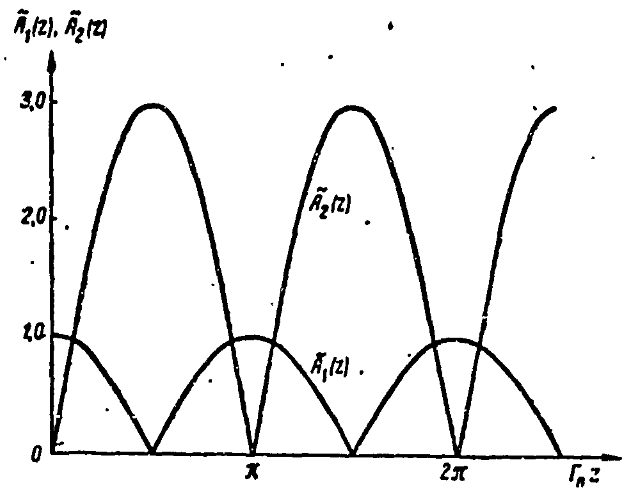

Fig. 4-4. Graphs of the change in space of amplitudes $\tilde{A}_1(z)=\frac{A_1}{A_{10}}$; $\tilde{A}_2(z)=\frac{A_2}{A_{10}}$ for a parametric interaction of the form $\omega_1+\omega_H=\omega_2$; $k_1+k_H=k_2$ with boundary conditions $A_{10}\neq 0$; $A_{20}=0$.

In those cases when the energy passes from the wave of the smaller frequency to a wave of greater frequency, the greater the pumping accomplishes positive work, the larger the ratio $\frac{\omega_2}{\omega_1}$ [see (4.16)]. With reverse transition, on the contrary, the wave of pumping absorbs part of the energy - it accomplishes negative work. Therefore, an interaction of the type (4.2) can be used in nonlinear optics for amplification with simultaneous conversion of the frequency upwards.

Thus, just as for amplification with high-frequency pumping,

the condition of synchronism for the examined interaction can be carried out in a uniaxial crystal. Here waves at frequencies ω_1 and ω_H – extraordinary (see also Fig. 4-1).

2.3. On Parametric Amplification with Low-Frequency Pumping

For an interaction of the form (4.3) the field in the medium should be presented in the form

$$\mathbf{E} = \mathbf{e}_H A_H \exp i(\omega_H t - \mathbf{k}_H \mathbf{r}) + \sum_{l=1}^{4} \mathbf{e}_l A_l(\mu \mathbf{r}) \exp i(\omega_l t - \mathbf{k}_l \mathbf{r}) = \\ = \mathbf{E}_H + \sum_{l=1}^{4} \mathbf{E}_l. \tag{4.17}$$

For simplicity we will consider also that together with (4.3) conditions of synchronism of the form

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_H;\ \mathbf{k}_1 + \mathbf{k}_H = \mathbf{k}_3;\ \mathbf{k}_2 + \mathbf{k}_H = \mathbf{k}_4. \tag{4.18}$$

are fulfilled.

Then, disregarding losses in the medium, for complex amplitudes A_1-A_4 we will obtain the truncated equations

$$k_1 \cos \widehat{\mathbf{k}_1\mathbf{s}_1} \cos \widehat{\mathbf{s}_1\mathbf{z}_0} \frac{dA_1}{dz} + i\frac{2\pi}{c^2}\omega_1^2 \left(\mathbf{e}_1 \hat{\chi}^{\omega_H - \omega_2} \mathbf{e}_H \mathbf{e}_2\right) A_H A_2^* + \\ + i\frac{2\pi}{c^2}\omega_1^2 \left(\mathbf{e}_1 \hat{\chi}^{\omega_3 - \omega_H} \mathbf{e}_3 \mathbf{e}_H\right) A_3 A_H^* = 0; \tag{4.19a}$$

$$k_2 \cos \widehat{\mathbf{k}_2\mathbf{s}_2} \cos \widehat{\mathbf{s}_2\mathbf{z}_0} \frac{dA_2}{dz} + i\frac{2\pi}{c^2}\omega_2^2 \left(\mathbf{e}_2 \hat{\chi}^{\omega_H - \omega_1} \mathbf{e}_H \mathbf{e}_1\right) A_H A_1^* + \\ + i\frac{2\pi}{c^2}\omega_2^2 \left(\mathbf{e}_2 \hat{\chi}^{\omega_4 - \omega_H} \mathbf{e}_4 \mathbf{e}_H\right) A_4 A_H^* = 0; \tag{4.19b}$$

$$k_3 \cos \widehat{\mathbf{k}_3\mathbf{s}_3} \cos \widehat{\mathbf{s}_3\mathbf{z}_0} \frac{dA_3}{dz} + i\frac{2\pi}{c^2}\omega_3^2 \left(\mathbf{e}_3 \hat{\chi}^{\omega_H + \omega_1} \mathbf{e}_H \mathbf{e}_1\right) A_H A_1 = 0; \tag{4.19c}$$

$$k_4 \cos \widehat{\mathbf{k}_4\mathbf{s}_4} \cos \widehat{\mathbf{s}_4\mathbf{z}_0} \frac{dA_4}{dz} + i\frac{2\pi}{c^2}\omega_4^2 \left(\mathbf{e}_4 \hat{\chi}^{\omega_H + \omega_2} \mathbf{e}_H \mathbf{e}_2\right) A_H A_2 = 0. \tag{4.19d}$$

In accordance with what has been said in Chapter II solution to the system (4.19) can be represented in the form

$$A_l(\mu z) = A_{l_0} \exp \Gamma z, \quad l = 1, 2, 3, 4. \tag{4.20}$$

Substituting (4.20) into (4.19), we arrive at the dispersion equation of the fourth order for the propagation constant Γ. Numerical analysis of the dispersion equation (see, for example [60]) shows that the purely exponential accretion of amplitudes A_1 and A_4 proves to be impossible; the interaction of the waves has a character of spatial beats, and the amplitudes of maxima are increased with an increase in z.

The appearance of the exponential by growing waves with consecutive three-frequency interactions is possible if dispersion properties of the medium allow only the fulfillment of the first two equalities of (4.18) – only one of the waves "sum" frequencies ω_3 and ω_4 can coherently interact with the remaining waves. In order to be convinced in this, let us assume in (4.19) $A_4 \equiv 0$. Then system (4.19) can be presented in the form (the amplitude of the wave of pumping, not limiting the community, can be considered real)

$$\frac{dA_1}{dz} + i(\sigma_1' A_2^* + \sigma_1^* A_3) = 0; \tag{4.21a}$$

$$\frac{dA_2^*}{dz} - i\sigma_2 A_1 = 0; \tag{4.21b}$$

$$\frac{dA_3}{dz} + i\sigma_3 A_1 = 0; \tag{4.21c}$$

$$\sigma_1^{\circ} = \frac{2\pi\omega_1^2 \left(\mathbf{e}_1 \hat{\chi}^{\omega_н - \omega_2} \mathbf{e}_н \mathbf{e}_2 \right) A_н}{c^2 k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}}; \quad \sigma_1^{*} = \frac{2\pi\omega_1^2 \left(\mathbf{e}_1 \hat{\chi}^{\omega_3 - \omega_н} \mathbf{e}_3 \mathbf{e}_н \right) A_н}{c^2 k_1 \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}};$$

$$\sigma_2 = \frac{2\pi\omega_2^2 \left(\mathbf{e}_2 \hat{\chi}^{\omega_н - \omega_1} \mathbf{e}_н \mathbf{e}_1 \right) A_н}{c^2 k_2 \cos \widehat{\mathbf{k}_2 \mathbf{s}_2} \cos \widehat{\mathbf{s}_2 \mathbf{z}_0}}; \quad \sigma_3 = \frac{2\pi\omega_3^2 \left(\mathbf{e}_3 \hat{\chi}^{\omega_н + \omega_1} \mathbf{e}_н \mathbf{e}_1 \right) A_н}{c^2 k_3 \cos \widehat{\mathbf{k}_3 \mathbf{s}_3} \cos \widehat{\mathbf{s}_3 \mathbf{z}_0}}.$$

Differentiating (4.21a) with respect to z and substituting into the

obtained expression (4.21b) and (4.21c), we arrive at the differential second-order equation for A_1:

$$\frac{d^2 A_1}{dz^2} = (\sigma_1' \sigma_2 - \sigma_1'' \sigma_3) A_1 \tag{4.22}$$

and, consequently, when $\sigma_1' \sigma_2 > \sigma_1'' \sigma_3$ the solution of (4.22) has the form

$$A_1 = a_1 e^{\Gamma z} + b_1 e^{-\Gamma z}. \tag{4.23}$$

Substituting (4.23) into (4.21c), we are convinced that the wave of frequency ω_3, exceeding the frequency of pumping grows exponentially. The unbounded accretion of amplitudes of the parametrically interacting waves, of course, cannot take place; in the quadratic medium limitation of the amplitude (saturation of the parametric amplifier of the traveling wave) occurs owing to the reverse reaction of growing waves on the wave of pumping (let us remember that formulas (4.9) (4.23) are obtained in the approximation of the assigned field). Let us turn to the investigation of the effects of saturation; the greatest interest in such investigation is for conditions of amplification with high-frequency pumping.

§ 3. Effects of Saturation with Parametric Amplification of Traveling Waves in a Quadratic Medium. A Tunable Parametric Light Generator

3.1. Conditions of Saturation of an Amplifier with High-Frequency Pumping

To investigate the effects of saturation in an amplifier with high-frequency pumping, besides equations one should consider (4.4) also equations describing the change in amplitude and phase of the wave of pumping. Truncated equations for real amplitudes and phases, which completely describe the three-frequency interaction in a quadratic medium, have the form:

$$\frac{dA_1}{dz} + \sigma_1 A_2 A_н \sin\Phi + \delta_1 A_1 = 0; \tag{4.24a}$$

$$\frac{dA_2}{dz} + \sigma_2 A_1 A_н \sin\Phi + \delta_2 A_2 = 0; \tag{4.24b}$$

$$\frac{dA_H}{dz}-\sigma_3 A_1 A_2 \sin\Phi+\delta_3 A_3=0; \qquad (4.24c)$$

$$\frac{d\Phi}{dz}+\Delta+\left(\sigma_1\frac{A_2 A_H}{A_1}+\sigma_2\frac{A_1 A_H}{A_2}-\sigma_3\frac{A_1 A_2}{A_H}\right)\cos\Phi=0. \qquad (4.24d)$$

Here the meaning of parameters σ_1, σ_2, σ_3, δ_1, δ_2, δ_3 and Δ is conventional, and $\Phi = \phi_1 + \phi_2 - \phi_H$. System (4.24) in general can be solved only numerically; when $\delta_1 = \delta_2 = \delta_3 = \delta$ (in particular, $\delta = 0$) the equations possess the first two integrals (see (2.43) and (2.43a) of § 3 of Chapter III): using the first integrals, it is possible to exclude, for example, variables A_2 and A_3 and obtain an equation describing the behavior of phase trajectories on the plane A_1^2, Φ:

$$\frac{d\Phi}{dA_1^2}=f(A_1^2,\Phi,\Delta). \qquad (4.25)$$

We will not conduct detailed analysis of (4.25) here: it basically is analogous to that conducted in § 3 of Chapter III and is carried out in work [157]. Here we will limit ourselves to consideration of the simplest $\Delta = 0$ and $\delta = 0$ for which the obtaining of analytic relations prove to be possible (see also [158]). We will consider that when $z = 0$

$$A_1(0)=A_{10};\ A_2(0)=0;\ A_H(0)=A_{H0};\ \Phi(0)=\Phi_0=-\frac{\pi}{2}. \qquad (4.26)$$

Motion with boundary conditions (4.26) is obviously the motion along the separatrix (see Chapter III), and therefore $\sin\Phi = \pm 1$; $\cos\Phi = 0$. From equations (4.24a, b) in examined case we have:

$$\frac{dA_1}{dA_2}=\frac{\sigma_1 A_2}{\sigma_2 A_1}$$

and, using (4.26),

$$A_1^2=\frac{\sigma_1}{\sigma_2}A_2^2+A_{10}^2. \qquad (4.27)$$

Similary we have

$$A_x^2 = -\frac{\sigma_2}{\sigma_3} A_2^2 + A_{x0}^2; \tag{4.28}$$

$$A_1^2 = -\frac{\sigma_1}{\sigma_2} (A_x^2 - A_{x0}^2) + A_{10}^2. \tag{4.29}$$

Substituting (42.7)-(4.28) into (4.24b), we obtain:

$$\frac{dA_2}{dz} = -\left[\sigma_1 \sigma_3 \left(\frac{\sigma_2}{\sigma_3} A_{x0}^2 - A_2^2\right)\left(\frac{\sigma_2}{\sigma_1} A_{10}^2 + A_2^2\right)\right]^{1/2}. \tag{4.30}$$

Introducing designations $w^2 = \frac{\sigma_2}{\sigma_3} A_{x0}^2$; $v^2 = \frac{\sigma_2}{\sigma_1} A_{10}^2$ and integrating (4.30), we obtain:

$$\int_0^{A_2} \frac{dA_2}{\sqrt{(w^2 - A_2^2)(v^2 - A_2^2)}} = -\sqrt{\sigma_1 \sigma_3} \int_0^z dz. \tag{4.31}$$

Introducing a new variable $w^2 - A_2^2 = w^2 y^2$, the integral in the left part of (4.31) can be reduced to an elliptic integral of the first kind. Then instead of (4.31) we have:

$$\frac{1}{(w^2+v^2)^{1/2}} \int_1^{\left[1-\frac{A_2^2}{w_2^2}\right]^{1/2}} \frac{dy}{\sqrt{(1-y^2)(1-k^2 y^2)}} = \sqrt{\sigma_1 \sigma_3}\, z. \tag{4.32}$$

ere

$$k^2 = \frac{w^2}{v^2+w^2} = \left(1 + \frac{\sigma_3}{\sigma_1} \frac{A_{10}^2}{A_{x0}^2}\right)^{-1}. \tag{4.33}$$

Turning the elliptic integral, it is possible to arrive, as is known, at the Jacobi elliptic functions.

From (4.32) for A_2 we have

$$A_3(z)=\sqrt{\frac{\sigma_3}{\sigma_3}}A_{H0}\cdot\mathrm{cn}\left(K+\frac{\sqrt{\sigma_1\sigma_3}}{k}A_{H0}z\right), \qquad (4.34a)$$

where

$$K=\int\limits_0^1\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}.$$

Using (4.34a), formulas describing the change in space of amplitudes A_1 and A_3 can be obtained;

$$A_1(z)=\sqrt{\frac{\sigma_1}{\sigma_3}}\cdot A_{H0}\cdot\frac{1}{k}\,\mathrm{dn}\left(K+\frac{\sqrt{\sigma_1\sigma_3}}{k}A_{H0}z\right]; \qquad (4.34b)$$

$$A_H(z)=A_{H0}\cdot\mathrm{sn}\left(K+\frac{\sqrt{\sigma_1\sigma_3}}{k}A_{H0}z\right). \qquad (4.34c)$$

Using (4.34), it is possible to construct graphs of the change in power fluxes along the z axis: $S_1=[\mathbf{E}_1\mathbf{H}_1^*]\mathbf{z}_0$; $S_2=[\mathbf{E}_2\mathbf{H}_2]\mathbf{z}_0$; $S_H=[\mathbf{E}_H\ \mathbf{H}_H]\mathbf{z}_0$. In Fig. 4-5 such graphs are constructed for boundary conditions (4.26). From the given curves it follows that the interaction of waves in the examined case has a character of spatial beats; the exponential growth of amplitudes A_1, A_2 at A_1, $A_2 \sim A_H$ is delayed, and amplitudes of waves of the signal and difference frequency reach a maximum and then start to decrease, transmit their energy to the wave of pumping. The period of spatial beats

$$\Lambda_0\simeq\frac{Kk}{\sqrt{\sigma_1\sigma_3}A_{H0}},$$

and maximum power amplification

$$\frac{S_{1\,\text{макс}}}{S_{10}}=1+\frac{\omega_1}{\omega_H}\frac{S_{H0}}{S_{10}} \qquad (4.35)$$

and, consequently, if $\frac{\omega_1}{\omega_H}$ is not to small, the efficiency of the amplifier can reach units and tens of percent. Losses and deviations from conditions of synchronism worsen characteristics of the amplifier.

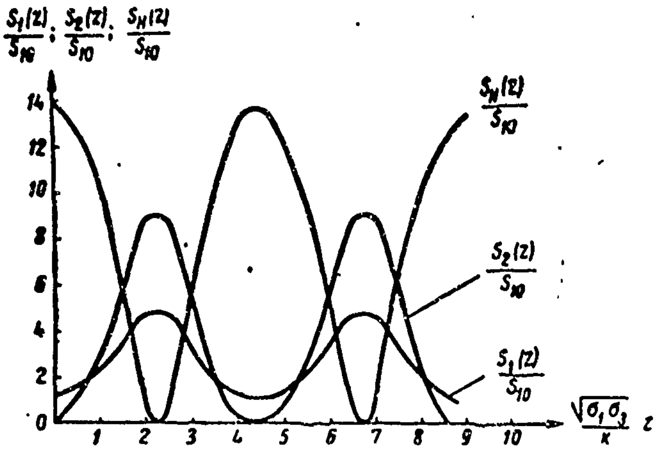

Fig. 4-5. Graphs of the change in space of powers of the signal $S_1(z)/S_{10}$, difference frequency $S_2(z)/S_{10}$, and pumping $S_H(z)/S_{10}$ with parametric amplification of traveling waves in a quadratic medium.

Just as in the theory of the generation of harmonics, the influence of the indicated factors is determined by values of the given parameters $\tilde{\delta} = \frac{\delta}{\sigma_1 A_{H0}}$; $\tilde{\Delta} = \frac{\Delta}{\sigma_1 A_{H0}}$. When $\tilde{\delta} < 1, \tilde{\Delta} < 1$ the character of the processes in the system qualitatively does not differ from the case $\tilde{\Delta} = \tilde{\delta} = 0$. Let us note, however, that when $\tilde{\delta} \neq 0$ together with the spatial beats there takes place a monotonic decrease in amplitudes of the interacting waves. At sufficiently large $\tilde{\delta}$ amplitudes A_1 and A_2 on the segment of change in z [0; ∞] have only one maximum each (see [157]).

The method stated above of the calculation can be used and during the analysis of interaction of the type (4.2)[1] in those cases when and here it impossible to be limited to concepts about the assigned field of pumping. Let us note, however, that the absence of exponential growing waves in the last case makes this analysis less urgent.

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[1]Let us also note that this method is completely applied to the problem on the generation of the second harmonic in the two-dimensional medium (see (3.19a)). It is not difficult to see that here when $A_1^{(1)}(0) \neq A_1^{2}(0)$ the spatial beats will take place when $\Delta = 0$.

Investigation of conditions of saturation of the parametric amplifier with high-frequency pumping is of interest not only from the point of view of the problem on the calculation of its maximum output power but esp· ally in connection with the problem of parametric generat·on of electromagnetic oscillations. Acutally, it is easy to see that if being under the influence of an intense wave of pumping quadratic medium is placed in a resonator, possessing sufficient high quality, in the medium oscillations at frequencies ω_1 and ω_2 can be self-excited. Such a generator represents special interest in the optical range (cm [63, 64, 144, 145]), inasmuch as at a fixed frequency ω_H in principle considerable returning of frequencies ω_1 and ω_2 is possible (let us recall that if we are distracted at present from dispersion properties of the medium, the only condition superimposed on frequencies ω_1 and ω_2 is the condition (4.1).

It is necessary to note that parametric generators of the indicated type are investigated in detail in the radio-frequency band (there we usually call them two-circuit parametric generators, see for example, [148]). However, in optics such generators possess a number of peculiarities, and we will turn to a brief analysis of them.

3.2. Parametric Light Generator

A diagram of a tuned generator is shown in Fig. 4-6a. Falling here on the quadratic crystal is the wave of pumping E_H, freely penetrating the latter. Directions l' and l'' are selected in such a way that waves of frequencies ω_1 and ω_2, propagating in the indicated directions, can coherently interact with the wave of pumping

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_H$$

and, consequently, in virtue of that discussed in § 2 of this chapter

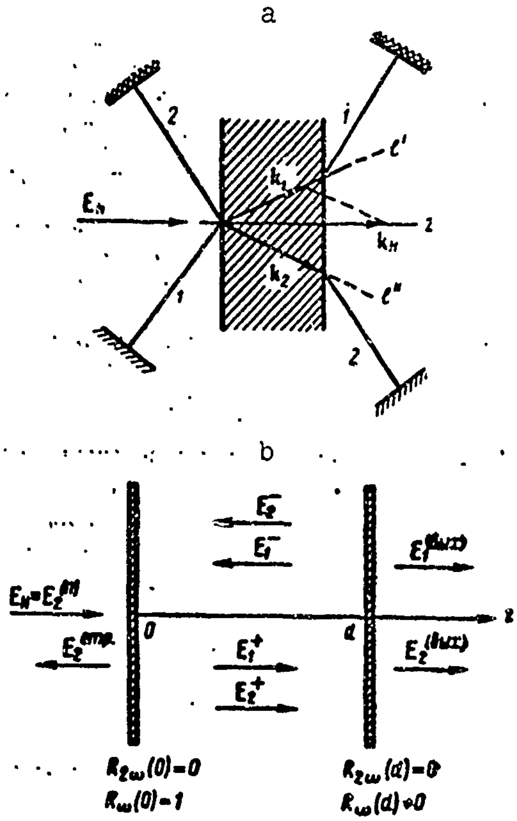

Fig. 4-6. Diagrams of parametric light generators: a) tuned double-resonator generator; b) monoresonator (degenerated) parametric generator

when $A_{1,2} \ll A_H$, $A_1 \sim e^{\Gamma_0 z}$, $A_2^* \sim e^{\Gamma_0 z}$. If now in direction of rays 1, 2 (let us recall that waves at frequencies $\omega_{1,2}$ are ordinary), which emerge from the crystal, we install mirrors (see Fig. 4-6a), in the system there appears positive feedback, and self-excitation of the oscillations becomes possible.

Values of frequencies of self-excited oscillations are determined, obviously, by the position of the mirrors.

We will not discuss in detail the analysis of factors determining the range of smooth retuning of the generator; it is easy to see that it is connected, first of all, with linear dispersion properties of the quadratic medium (see [63]).

Let us turn to the investigation of the process of excitation of parametric oscillations. Let us consider the most simple variant of the parametric generator - the so-called degenerated parametric generator (see also [144]-[145]), in which

$$\omega_1 = \omega_2 = \omega = \frac{\omega_H}{2}. \quad (4.36)$$

Self-excitation of degenerated parametric oscillations is possible, obviously, in a one-dimensional resonator of the type Fabry-Perot[1], which contains the quadratic medium oriented in such a way that the phase speed of the ordinary wave of frequency ω is equal to the phase speed of the extraordinary wave of frequency $\omega_H = 2\omega$ in a direction perpendicular to mirrors of the resonator. A diagram of such a generator is shown in Fig. 4-6b. Here a plane wave of pumping is in incident on the Fabry-Perot resonator:

$$\mathbf{E}_H = \mathbf{e}_H A_{H0} \exp i(\omega_H t - \mathbf{k}_H \mathbf{r}). \quad (4.37)$$

It is assumed that the resonator is transparent for the wave of pumping; reflection factors with respect to amplitude at frequency $\omega_H = 2\omega$

$$R_{2\omega}(0) = R_{2\omega}(d) = 0. \quad (4.38a)$$

If the nonlinear medium occurring in the resonator is oriented in such a way that conditions of synchronism are fulfilled for frequencies (4.36) and reflection factors at frequency ω

$$R_\omega(0) \neq 0; \quad R_\omega(d) \neq 0 \quad (4.38b)$$

---

[1]In a one-dimensional resonator nondegenerate oscillations $\omega_1 + \omega_2 = \omega_H$ are possible, of course (see Fig. 4-1). However, here a change in the generated frequencies is possible only with a change in the direction of the wave vector of pumping k_H.

the thermal fluctuations inevitably present in the resonator can cause self-excitation of oscillations at frequency ω[1]. Here the field in the resonator can be represented in the form of the superposition of direct (all values pertaining to them will be noted by a (+) sign) and return (-) waves:

$$\mathbf{E}^{+} = \mathbf{e}_2^{+} A_2^{+}(\mu z) \exp i(2\omega t - k_2 z) + \mathbf{e}_1^{+} A_1^{+}(\mu z) \exp i(\omega t - k_1 z). \quad (4.39)$$

$$\mathbf{E}^{-} = \mathbf{e}_2^{-} A_2^{-}(\mu z) \exp i(2\omega t + k_2 z) + \mathbf{e}_1^{-} A_1^{-}(\mu z) \exp i(\omega t + k_1 z). \quad (4.40)$$

Values referring to the field of pumping are noted here by subscript 2(2ω), and to the field of parametric oscillations 1(1ω). It is important to emphasize that although relationship (4.38a) takes place, in the field of the return wave, especially at large A_1^-, there are inevitably present frequency oscillations 2ω – the return wave of the parametrically excited oscillations generates a second harmonic Therefore, in the examined diagram there always exists a "reflected" wave at frequency 2ω, $E_{(\text{отр})}$, which propagates in the direction of the generator of pumping.

If one were to be interested not only in steady-state oscillations of the parametric generator but also transition processes, resolution of problem can be obtained with the help of the procedure discussed in § 5 of Chapter III. The process of excitation of parametric oscillations can be presented as a sequence of steps in each of which the interaction of the waves is described by equations of the type(3.20). Then the initial equations, as in the problem on the resonator frequency doubler, here are equations (3.65) which must be solved with boundary conditions (compare 3.66).

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[1]Here it is appropriate to pay attention to the important distinction of the problem on the parametric generator (generator of subharmonic) from the problem examined in Chapter III on the generator of the second harmonic. If in the last case, the value of initial amplitude A_{20} is immaterial, the process of generation of the second harmonic proceeds when $A_{20} = 0$, parametric excitation is possible only at an initial amplitude of the subharmonic different from zero (or in the presence of a side force having components at a frequency of the order of frequency of the subharmonic).

$$\left.\begin{aligned} A^{+}_{1(N)}(0) &= R_{\infty}(0)\cdot A^{-}_{1(N-1)}(0) \\ A^{-}_{1(N)}(d) &= R_{\infty}(d)\cdot A^{+}_{1(N-1)}(d) \end{aligned}\right\} \tag{4.41a}$$

$$\varphi^{\pm}_{1(N)} = \varphi^{\mp}_{1(N-1)} + \pi \tag{4.41b}$$

$$A^{+}_{2(N)}(0) = A_{\text{н0}};\quad A^{-}_{2(N)}(d) = 0 \tag{4.41c}$$

N – as before, the number of the step. In accordance with that mentioned above on the role of initial conditions in the examined problem, the following certainly should be

$$A^{+}_{1(1)}(0) \neq 0: \tag{4.42}$$

the initial amplitude of the direct wave of the subharmonic for the first step should be different from zero[1].

In the analysis of equations (3.65) we will consider that the condition of synchronism is fulfilled exactly ($\Delta = 0$), and the ratio of the initial amplitude of the subharmonic to the amplitude of pumping is small

$$\frac{A^{+}_{1(1)}(0)}{A_{\text{н0}}} \ll 1. \tag{4.43}$$

We will consider also that losses in the medium are small ($\delta d \ll 1$) and the condition of the appearance of growing waves $A_{\text{н}} > A_{\text{пор}}$ where $A_{\text{пор}}$ is determined by relation (4.7) is fulfilled with a reserve, and $\Gamma \simeq \Gamma_0$.

Under the assumptions made an analysis of the process of establishing parametric oscillations can be conducted, by using only amplitude equations (3.65) – motion is accomplished along a

---

[1]If quantity $A^{+}_{1(1)}$ (0) has a fluctuating origin, into equations (3.65) there must be introduced, in general, fluctuating side forces (see § 4, Chapter II). If one is interested, however, in fluctuations of the amplitude and phase of parametric oscillations, calculation of side forces is equivalent to the calculation of the initial amplitude different from zero of the direct wave of the subharmonic.

trajector very close to the separatrix, phases Φ^+ and Φ^- on every step are constant, and their changes from step to step are connected only with jumps in the phase of the subharmonic on mirrows. Thus, instead of (3.65) we have:

$$\frac{dA_1^\pm}{dz} + \sigma_1 A_1^\pm A_2^\pm \sin\Phi^\pm = 0; \qquad (4.44a)$$

$$\frac{dA_2^\pm}{dz} - \sigma_2 (A_1^\pm)^2 \sin\Phi^\pm = 0 \qquad (4.44b)$$

(when δd << 1 and $A_H >> A_{\text{пор}}$ the distributed losses in the medium can be considered due to the appropriate correction of values R_ω).

We will turn, first of all, to the conclusion of conditions of parametric excitation of the oscillations. If (4.43) takes place, calculacion can be carried out in the approximation of the assigned field $A_2^+ = A_{HO}$, $A_2^- \equiv 0$. Considering $\sin \Phi+ = -1$ (see the phase plane of Fig. 3-6a) we have from (4.44a)

$$A_{1(N)}^+(z) = A_{1(N)}^+(0) \cdot \exp \sigma_1 A_{HO} z. \qquad (4.45)$$

At the fixed point of the resonator oscillations of the subharmonic will grow with time if the increase in the amplitude on the N-step exceeds the loss to radiation through the mirror, i.e., if $A_{1(N+2)}^+(0) > A_{1(N)}^+(0)$. In the approximation of the assigned field $A_{1(N+2)}^+(0) = A_{1(N)}^+(d) \cdot R_\omega(0) \cdot R_\omega(d)$. Using (4.45), the condition of self-exitation can be rewritten in the form

$$R_\omega(0) \cdot R_\omega(d) > e^{-\sigma_1 A_{HO} d} \qquad (4.46)$$

For $\sigma_1 A_{HO} d<1$, instead of (4.46), it is possible to use a more graphic formula

$$\frac{2\pi\omega^2 \left(e_1 p^{2\omega-\omega} \right) A_{HO}}{c^2 k_1^2} > \frac{1}{Q} \qquad (4.47)$$

(for the ordinary wave of the subharmonic in the examined case), where $Q=\frac{k_1 d}{1-R_\omega(0)\cdot R_\omega(d)}$ – high quality of the Fabry-Perot resonator. The term standing in the left side of inequality (4.47) can be called the effective modulation factor of the dielectric constant of the quadratic medium M; here condition (4.47) has the same form that of the corresponding condition of excitation of the parametric generator with lumped parameters [147][1].

It is not difficult to show that for the double-resonator generator, instead of (4.47) $M>\frac{1}{\sqrt{Q_1Q_2}}$. If the process of establishing oscillations in parametric generator is described by equations (4.44), the change in amplitudes of pumping and subharmonic on every step can be calculated by using solutions of the type (3.36) and (3.37).

---

[1]Thus, in the theory of parametric generation in the distributed medium, in contrast to the theory of diagrams with lumped parameters, two conditions of instability appear: the condition of instability in space (4.6) and in time (4.47).

We have:

$$A^{+}_{1(N)}(d) = \frac{A^{+}_{1(N)}(0)}{\sqrt{1-\left[\frac{A_{H0}}{A_{0N}}\right]^2}} \operatorname{sech} \sigma_1 A_{0N}(z_{0N}-d). \qquad (4.48)$$

$$A^{+}_{2(N)}(d) = A_{0N} \cdot \operatorname{th} \sigma_1 A_{0N}(z_{0N}-d). \qquad (4.49)$$

Here

$$A_{0N} = \sqrt{A^2_{H0} + \frac{\sigma_2}{\sigma_1}[A^{+}_{1(N)}(0)]^2}. \qquad (4.50)$$

$$z_{0N} = \frac{1}{\sigma_1 A_{0N}} \cdot \operatorname{arth} \frac{A_{H0}}{A_{0N}}. \qquad (4.51)$$

Quantity $A^{+}_{1(N)}(0)$ can be calculated in terms of $A^{+}_{1(N-2)}(d)$; here one should consider that part of the energy of the subharmonic is expended due to radiation through mirrors and owing to generation of the wave of frequency 2ω (second harmonic of parametric oscillations). The last process is described by formula (see (3.36)), and the boundary condition (4.41c)

$$A^{-}_{2(N)}(z) = \sqrt{\frac{\sigma_2}{\sigma_1}} A^{+}_{1(N-1)}(d) \cdot R_{\omega}(d) \cdot \operatorname{th} \times \\ \times \left[\sqrt{\sigma_1 \sigma_2} A^{+}_{1(N-1)} R_{\omega}(d) \cdot (d-z)\right], \qquad (4.52)$$

and a decrease in the amplitude of the subharmonic $A^{-}_{1}(z)$ occurs according to the law of the hyperbolic secant (3.37), so that:

$$A^{+}_{1N}(0) = A^{+}_{1(1)}(0) + R_{\omega}(0) R_{\omega}(d) \cdot A^{+}_{1(N-2)}(d) \cdot \operatorname{sech} \times \\ \times \left[\sqrt{\sigma_1 \sigma_2} R_{\omega}(d) \cdot A^{+}_{1(N-2)}(d) \cdot d\right]. \qquad (4.53)$$

Using (4.48)-(4.53), one can determine law of the establishment of parametric oscillations.

Results of the appropriate calculation are given on the graph of Fig. 4-7. Illustrated here are laws of the change in relative amplitudes $\tilde{A}_2 = \frac{A^{+}_{2(N)}(d)}{A_{H0}}$ and $\tilde{A}_1 = \frac{A^{+}_{1(N)}(d)}{A_{H0}}$ as a function of the number of

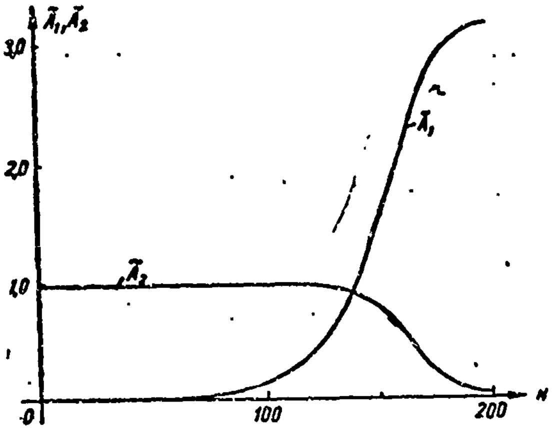

Fig. 4-7. Process of establishing the amplitude of steady-state oscillations in the degenerator. Plotted along the axis of the abscissas is the number of reflections in the resonator N, and along the axis of the ordinates – reduced amplitudes of pumping $\tilde{A}_2 = \frac{A^+_{2(N)}(d)}{A_{H0}}$ and subharmonic $\tilde{A}_1 = \frac{A^+_{1(N)}(d)}{A_{H0}}$ at the outlet mirror of the resonator. The parameter $A_{H0}\sigma_1 d = 0,1$; $R_\omega(0)=0,99$; $R_\omega(d) = 0,99$.

reflections N.[1] As follows from the given graph, the efficiency of the parametric generator can be sufficiently high. Using (4.48)-(4.53), it is possible to obtain calculation relationships which allow determining the amplitude of stead-state parametric oscillations.

Actually, in the steady-state operation

$$A^+_{1(N)}(d) = A^+_{1(N-2)}(d) = A_{1y}(d), \qquad (4.54)$$

whence, using (4.48) and (4.53), the transcendental equation for A_{1y}

[1]Calculation of the process of setting when $\Delta \neq 0$ shows that the fulfillment of conditions of self-excitation with a growth in Δ becomes all the more difficult; the efficiency of the parametric generator with a growth in Δ decreases, and the process of establishing oscillations has an oscillator character (compare Fig. 3-17a).

can be obtained. The solution of this transcendental equation can be obtained graphically. However, in the case when losses in the nonlinear medium are absent and the value of the parameter $\sigma_1 A_{\text{но}} d \ll 1$, it is possible to obtain (see also [145]) the approximate expression for the stationary amplitude $A_{1y}(d)$. For this one should use the energy considerations. In the steady-state operation, if losses in the resonator are connected only with radiation, we have:

$$[\mathbf{E}_2^{\text{п}} \cdot \mathbf{H}_2^{\text{п}}] = [\mathbf{E}_2^{\text{отр}} \cdot \mathbf{H}_2^{\text{отр}}] + [\mathbf{E}_2^{\text{вых}} \cdot \mathbf{H}_2^{\text{вых}}] + 2\,[\mathbf{E}_1^{\text{вых}} \cdot \mathbf{H}_1^{\text{вых}}]. \quad (4.55)$$

Here, for simplicity, $R_\omega(0)=1$ is accepted (energy of the subharmonic emerges from the resonator only through the right mirror).

$$A_{\text{но}}^2 - [A_2^{\text{отр}}]^2 - [A_2^{\text{вых}}]^2 = 2\left(1 - R_\omega^2(d)\right) A_{1y}^2(d). \quad (4.56)$$

Although in general, for the calculation of $A_2^{\text{отр}}$ and $A_2^{\text{вых}}$ one should use formulas of the form (4.49) and (4.52), and for $\sigma_1 A_{\text{но}} d \ll 1$ and $R_\omega(d) \simeq 1$ it is possible to simplify the problem, assuming that in the steady state of the amplitude of the direct and return wave the subharmonics in the resonator do not depend on coordinate z and are approximately equal to each other, i.e.,

$$A_{1y}^{+}(z) \cong A_{1y}^{-}(z) \simeq A_{1y}^{+}(d). \quad (4.57)$$

Using (4.57) and boundary conditions (4.41c), as a result of the integration of (4.44b) we obtain:

$$A_2^{\text{вых}} = A_2^{+}(\mathrm{d}) = A_{\text{но}} - \sigma_2 \left(A_{1y}^{+}\right)^2 d. \quad (4.58)$$

$$A_2^{\text{отр}} = \sigma_2 \left(A_{1y}^{-}\right)^2 \mathrm{d} = \sigma_2 \left(A_{1y}^{+}\right)^2 \mathrm{d}. \quad (4.59)$$

Substituting (4.58)-(4.59) into (4.56), we obtain:

$$[A_{1y}]^2 = \frac{1}{\sigma_2 d}\left(A_{\text{но}} - \frac{1 - R_\omega^2(d)}{\sigma_2 d}\right). \quad (4.60)$$

According to (4.60) the amplitude of the subharmonic does not turn into infinity when $R_\omega(d)=1$: losses in energy occur due to the generation of the wave of double frequency by the return wave of

subharmonic $\mathbf{E}_1^-$. This (only for the return wave) is explained by the nonmonotonic dependence of amplitude A_{1y}^+ on parameter $\sigma_2 d$ (let us recall that the point with coordinates $A_2=0$ and $A_1\neq 0$ on the phase plane of Fig. 3-6a is not special).

Using (4.60), one can determine the amplitudes $A_2^{вых}$ and $A_2^{отр}$

$$A_2^{вых} = \frac{1-R_\infty^2(d)}{\sigma_2 d}. \tag{4.61}$$

$$A_2^{отр} = A_{н0} - \frac{1-R_\infty^2(d)}{\sigma_2 d}. \tag{4.62}$$

From (4.61) there follows an important conclusion – the amplitude of the wave of pumping at the outlet of the parametrically excited optical resonator does not depend on the amplitude of pumping at the input $A_{н0}$. The latter means that the parametric generator is simultaneously a limiter of the amplitude of oscillations of pumping; this circumstance was noted by Siegman [145].

The examined models of the parametric generators are, of course, the simplest. In principle, by introducing resonance elements into the configuration of the amplifier with low-frequency pumping (see § 2 this chapter), self-excitation of oscillations can be obtained at the frequencies exceeding the frequency of pumping. There can be definite interest also in the parametric generator in which waves of the subharmonic and pumping are exchanged by energies on the border of the nonlinear medium.

§ 4. Nonresonant Parametric Amplification in a Cubic Medium

4.1. Parametric Amplification in a Cubic Medium in the Presence of a Static Field

Inasmuch as in the presence of a static field in a cubic medium three-frequency interactions are solved, relationships between frequencies of parametrically interacting waves have the same form as that for the case of the quadratic medium [see (4.1)-(4.3)]. At the same time, in contrast to the quadratic medium, an essential

role here can be played by incoherent nonlinear effects connected with nonlinear corrections to the dielectric constant of the cubic medium. An especially essential role of incoherent effects appears in conditions close to saturation of the amplifier; a change in phase speeds of interacting waves and absorption due to the correction to the dielectric constant can cause a decrease in the amplification.

Below we will explain that stated in the example of an amplifier with high-frequency pumping. In the approximation of the assigned field $A_н = \text{const}$, the amplification of "weak" waves with frequencies $\omega_{1,2}$, $\omega_1 + \omega_2 = \omega_н$, $A_н$ $A_{1,2}$ is described by equations (just as in Chapter III, let us assume $\omega_1, \omega_2, \omega_н \ll \omega_{0i}$)

$$\frac{dA_1}{dz} + \sigma_1 A_н A_2 \sin\Phi + \delta_1 A_1 = 0; \tag{4.63a}$$

$$\frac{dA_2}{dz} + \sigma_2 A_н A_1 \sin\Phi + \delta_2 A_2 = 0; \tag{4.63b}$$

$$\frac{d\Phi}{dz} + \Delta^{(л)} + \Delta^{(н.л)} + \left(\sigma_1 \frac{A_2}{A_1} + \sigma_2 \frac{A_1}{A_2}\right) A_н \cos\Phi = 0. \tag{4.63c}$$

Designations in equations (4.63) are standard $\Phi = \varphi_1 + \varphi_2$; the nonlinear detuning (see § 4 Chapter III)

$$\Delta^{(н.л)} = \gamma_1 A_{н0}^2 + \gamma_2 A_1^2 + \gamma_3 A_2^2. \tag{4.64}$$

Let us note, first of all, that the presence of nonlinear detuning considerably affects the form of the region of parametric amplification (more accurately, the region of "instability in space" of the zero state of equilibrium). In Fig. 4-8 there are constructed regions of instability of the state $A_1 = A_2 = 0$ in the space for cased $\delta_{1,2} = 0$ and $\delta_{1,2} \neq 0$ and $\gamma_{1,2,3} < 0$ (the method of their calculation is absolutely analogous to that discussed in § 2 of this chapter). It is clear that in contrast to the case of the purely quadratic medium, the regions of instability are now no longer symmetric relative to the straight line $\Delta^{(л)} = 0$; the latter is fully evident, inasmuch as with the growth in $\Delta^{(л)} > 0$ and $\gamma_1 < 0$ the nonlinear detuning $\Delta_0^{(н.л)} = \gamma_1 A_{н0}^2$ compensates the linear at large valves of the amplitude of pumping. However, the most important distinction of equations (4.36) from

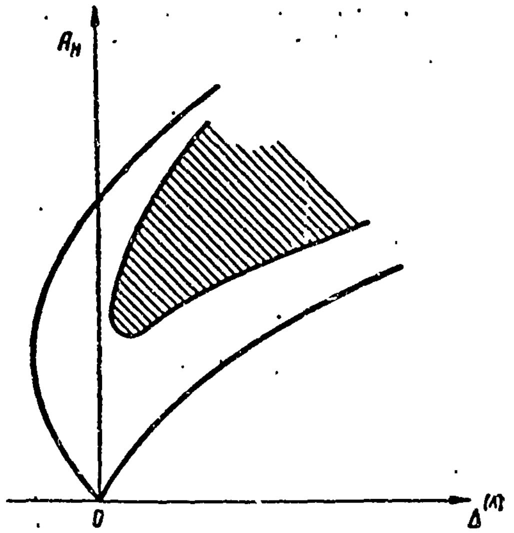

Fig. 4-8. Regions of parametric amplification of traveling waves a cubic medium occurring under influence of a static field in coordinates $\Delta^{(3)}, A_H(\gamma_{1,2,3}<0)$.

analogous equations (4.4), which correspond to amplification in a quadratic medium, is the fact that for equations (4.63) there are steady-state solutions, i.e., saturation of the amplifier can occur in the assigned field of pumping. In order to be convinced of this, let us assume in (4.63) $\frac{dA_1}{dz} = \frac{dA_2}{dz} = \frac{d\Phi}{dz} = 0.$

Then from (4.63a) and (4.63b) we have:

$$\frac{A_{1y}^2}{A_{2y}^2} = \frac{\sigma_1 \delta_2}{\sigma_2 \delta_1}. \qquad (4.65)$$

Using (4.65), one can determine the steady-state value of phase Φ:

$$A_{H0} \sin \Phi_y = -\sqrt{\frac{\delta_1 \delta_2}{\sigma_1 \sigma_2}}. \qquad (4.66)$$

Using (9) and (11) for the steady-state values of amplitudes, we obtain:

$$A_{1y} = \sqrt{\frac{2}{3\xi} \left\{\Delta^{(3)} + \gamma_1 A_{H0}^2 \mp (\delta_1 + \delta_2)\right\} \sqrt{\frac{\sigma_1 \sigma_2}{\delta_1 \delta_2} A_{H0}^2 - 1}}. \qquad (4.67)$$

$$A_{2y} = \sqrt{\frac{\xi}{\eta}} A_{1y}. \qquad (4.68)$$

Here

$$\xi = \gamma_1'\left(\frac{1}{2} + \frac{\sigma_2\delta_1}{\sigma_1\delta_2}\right) + \gamma_2'\left(1 + \frac{1}{2}\frac{\sigma_2\delta_1}{\sigma_1\delta_2}\right);$$
$$\eta = \gamma_1'\left(1 + \frac{1}{2}\frac{\sigma_1\delta_2}{\sigma_2\delta_2}\right) + \gamma_2'\left(\frac{1}{2} + \frac{\sigma_1\delta_2}{\sigma_2\delta_1}\right).$$

Here γ_1' and γ_2' are linear combinations of parameters γ_1, γ_2, and γ_3.

It is not difficult to understand the physical meaning of the obtained result. The part of the energy transmitted by the wave of pumping to waves of frequencies ω_1 and ω_2 depends on the value of phase $\Phi = \varphi_1 + \varphi_2$. At small amplitudes of A_1 and A_2 the value of phase Φ is determined by the detuning

$$\Delta_\Sigma^{(1)} = \Delta^{(\pi)} + \gamma_1 A_{\text{но}}^2,$$

and here $\sin\Phi \simeq -1$, and pumping with a reserve compensates losses in the medium. With a growth in amplitudes A_1 and A_2 the value of the nonlinear detuning is changed, and phase Φ departs from the value corresponding to the maximum release in energy of pumping to the amplified waves. In the steady state amplitudes of waves A_1 and A_2 do not depend on z, and phase Φ_y takes such a value that the wave of pumping accurately compensates losses in the medium (see (4.66)). When $\delta_{1,2}=0$, $\sin\Phi_y=0$. Peculiarities of the process of parametric amplification of traveling waves in a medium, the dielectric constant which depends on the amplitude of the wave, can be very visually illustrated if one were to turn to an examination of the phase plane of the amplifier. In the presence of damping, equations (4.63) yield to analysis on the phase plane in the case $\omega_1 = \omega_2 = \frac{\omega_н}{2}$ (so-called degenerated conditions of amplification).

Here instead of (4.63) we have ($A_1 = A_2 = A$, $\varphi_1 = \varphi_2 = \varphi$):

$$\frac{dA}{dz} + \sigma A_{\text{но}} A \sin 2\varphi + \delta A = 0. \tag{4.69}$$

$$\frac{d\varphi}{dz} + \Delta^{(\pi)} + \Delta^{(\text{нл})} + \sigma A_{\text{но}} \cos 2\varphi = 0. \tag{4.70}$$

$$\Delta^{(\text{нл})} = \gamma_1 A_{\text{но}}^2 + \gamma_2 A^2.$$

In accordance with (4.66)-(4.68) in the degenerated parametric amplifier

$$A_y = \sqrt{\frac{1}{\gamma_2}\left\{\Delta^{(\lambda)} + \gamma_1 A_{\text{но}}^2 \mp 2\delta\sqrt{\frac{\sigma^2 A_{\text{но}}^2}{\delta^2} - 1}\right\}} \quad (4.71)$$

and

$$\sin 2\varphi_y = -\sqrt{\frac{\delta^2}{\sigma^2 A_{\text{но}}^2}}. \quad (4.72)$$

From (4.72) it follows that in the degenerated parametric amplifier of the traveling wave four steady-state phases are possible (these phases are counted off from the phase of the wave of pumping); here only two prove to be stable.[1]

Figure 4-9 gives a phase plane corresponding to the system (4.69)-(4.70). Here, just as before, $X=A\sin\varphi$, $Y=A\cos\varphi$. An analysis of the structure of the phase plane can be conducted by the usual methods of the theory of oscillations. Singular points corresponding to the steady states here prove to be focuses. The character of the behavior of phase trajectories at great distances from the origin of the coordinates is easily established by calculating the tangent of the angle between the phase trajectory and radius vector:

$$\operatorname{tg}\alpha = A\frac{d\varphi}{dA} = \frac{\Delta^{(\lambda)} + \gamma_1 A_{\text{но}}^2 + \gamma_2 A^2 + \sigma A_{\text{но}}\cos 2\varphi}{\sigma A_{\text{но}}\sin 2\varphi + \delta}. \quad (4.73)$$

From (4.73) it is clear that when $A\to\infty$ the angle between the phase trajectory and radius vector $A^2=X^2+Y^2$ approaches 90° – at large A the phase trajectories twist around the origin of the coordinates and approach the circumferences in form.

In Fig. 4-9a the phase plane is constructed for that region where zero state of equilibrium is unstable – the amplitude of the

[1]The presence of two stable waves with different phases is of considerable interest from the point of view of applications (see [56], [133]).

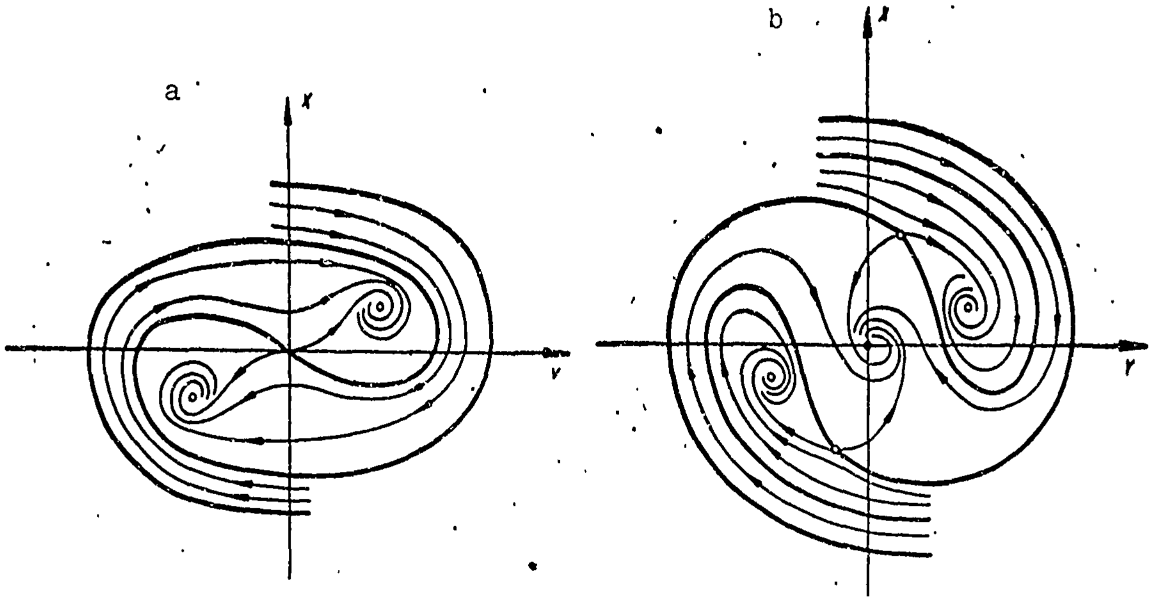

Fig. 4-9. Phase plane of a degenerated parametric amplifier of a traveling wave, the limitation of the amplitude of growing waves in which occurs due to the nonlinear corrections to the dielectric constant: a) the relationship between the linear and nonlinear detuning is such that there exist two stable states with amplitudes and phases unequal to zero distinguished by π; b) the zero state of equilibrium is stable; besides it there are still two stable states with amplitudes different from zero (region of stable amplification).

weak wave supplied to the input grows in space. (Here $|\Delta^{(x)} + \gamma_1 A_{\text{но}}^2| < \sqrt{\sigma^2 A_{\text{но}}^2 - \delta^2}$.) Behavior of the amplitude and phase of the amplified signal in space, which corresponds to the phase plane of Fig. 4-9a, is shown in Fig. 4-10. The parameter of curves here is served by the boundary phase $\varphi(0) = \varphi_0$. From the given curves it is clear that the process of amplification occurs here in such a way that at first the phase of the signal takes a value corresponding to the appearance of the negative absorption on the frequency of the signal. Here there starts the exponential growth of amplitude A. When $A \sim A_y$ the phase somewhat departs from the optimum – the amplifier is saturated, and the approach to conditions of saturation in the examined medium has an oscillatory character.

An interesting peculiarity of parametric amplification in the cubic medium is the presence here of very specific conditions of "stable" amplification appearing only at large amplitudes of the signal $A(0) > A_{\text{пр}}$. Actually, although at sufficiently large $\Delta^{(x)} > 0$ the

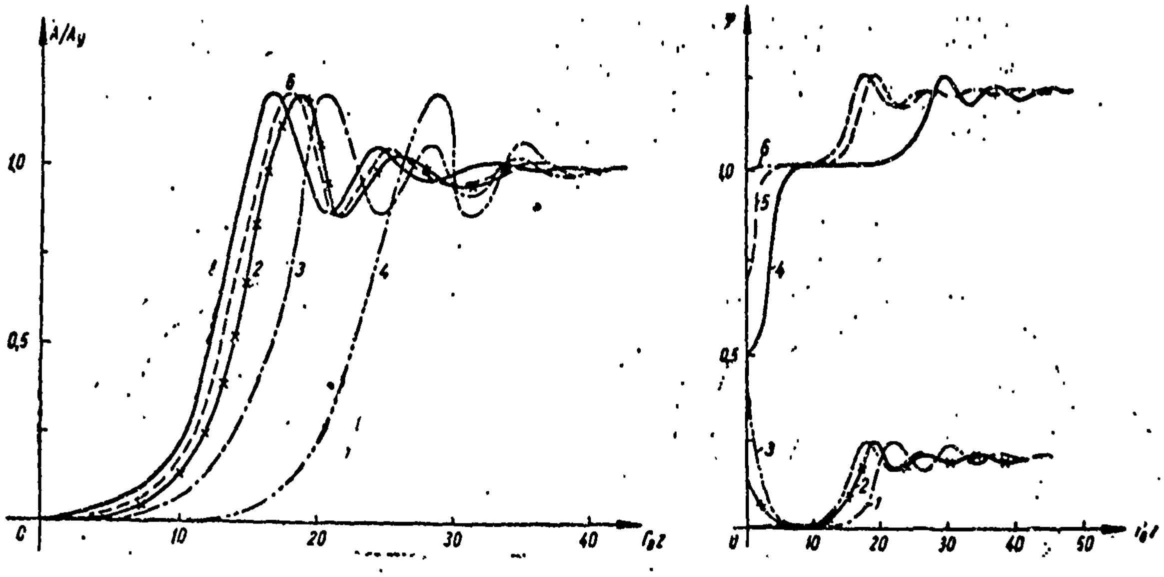

Fig. 4-10. Graphs of the dependence of amplitude $\tilde{A}_1 = \frac{A_1}{A_{1y}}$ and phase $\tilde{\varphi}_1 = \frac{\varphi}{\pi}$ of the growing wave on the coordinate z in a degenerated parametric amplifier of a traveling wave, in which the limitation of the amplitude occurs due to nonlinear corrections to the dielectric constant. The parameter of curves serves as the boundary phase $\hat{\varphi}_0$.

zero state of equilibrium is stable, here at the same time the stabilized amplitude A_y can be different from zero [see (4.71)]. The phase plane corresponding to conditions of "stable amplification" is constructed in Fig. 4-9b. (Here $\Delta^{(2)} + \gamma_1 A_H^2 > \sqrt{\sigma^2 A_H^2 - \delta^2}$).

4.2. Parametric Amplification with Four-Frequency Interactions

In the absence of a static field to the youngest of nonlinear interactions in a cubic medium is the four-frequency. Therefore, here the intense wave of pumping

$$\mathbf{E}_H = \mathbf{e}_H A_H \exp i(\omega_H t - \mathbf{k}_H \mathbf{r})$$

can transmit energy to weak waves the frequencies ω_1 and ω_2 and wave numbers $\mathbf{k}_1$ $\mathbf{k}_2$ of which satisfy relationships of the form:

$$\omega_1 + \omega_2 = 2\omega_н; \qquad (4.74a)$$
$$\mathbf{k}_1 + \mathbf{k}_2 = 2\mathbf{k}_н. \qquad (4.74b)$$

Let us note that if one were not interested in conditions of saturation, calculation of the amplification here can be conducted by proceeding from the model of the medium with variable parameters (see § 4 of Chapter II) – presenting the dielectric constant of the cubic medium in the form:

$$\hat{\varepsilon}(t, \omega, z) = \hat{e}_0(\omega) + \hat{M} e^{i(\Omega t - k_2 z)}, \qquad (4.75)$$

where the z axis is selected in the direction of vector $\mathbf{k}_н$, and $\Omega = 2\omega_н$, $k_2 = 2k_н$. Here the modulation factor of the dielectric constant $M \sim \theta A_н^2$ and, consequently, the factor of accretion Γ_0 is here proportional to the square, but not the first degree of the amplitude of pumping as takes place in the quadratic medium [see, for example, (4.8d)]. It is interesting to note that in the examined amplifier it is comparatively simple to satisfy the condition of synchronism (4.74b). Indeed, for conditions close to the degenerated $\omega_1 \simeq \omega_2 \simeq \frac{\Omega}{2} \simeq \omega_н$, the electromagnetic wave carrying out modulation of parameters of the cubic medium has a frequency close to frequencies of amplified waves.

We will not more specifically discuss the analysis of parametric interactions of the type (4.74) – the smallness of cubic nonlinearity in real optically transparent media makes, in any case at present, an experimental realization of these interactions difficult. Let us note also that interactions of the type (4.74) are not realized now and in the radio-frequency band (although in the theoretical work Fountana, Pantell and Smith [162] there was noted the possibility of parametric excitation of oscillations in the millimeter range, with the use of cubic nonlinearity of molecules of gas for $A_н \simeq 10^4$ V/cm and high quality of the resonator $Q \simeq 10^4$).

§ 5. Resonance Parametric Effects – Forced Combinational Scattering

Thus far in examining parametric effects, we originated essentially from the model of the nonlinear medium introduced in Chapter I

and described by equations (1.17) and (1.41). In the indicated model, in examining the nonlinear addition to potential energy there were considered only normal oscillations possessing a dipole moment different from zero and therefore directly connected with the electromagnetic field. At the same time, in the molecules symmetric oscillations not possessing the dipole moment and appearing therefore in absorption spectra are possible. In a linear approximation, symmetric oscillations of atoms (natural frequencies of these oscillations lie usually in the infrared range) and the electron oscillations determining the polarizability are accomplished independently of each other. Another situation takes place if one were to consider nonlinear terms in the expression for the potential energy of the molecule.

In order to explain what has been said, we will examine the isotropic medium, for example, liquid. Let us designate by x the normal coordinate of oscillations of atoms in a molecule of the examined medium and by y – the normal coordinate of oscillations of the electrons.

In the isotropic molecule the expression for potential energy, taking into account the younger nonlinear terms, has the form:

$$U=\frac{1}{2}Fx^2+\frac{1}{2}fy^2+\alpha_1x^3+\alpha_2x^2y+\alpha xy^2+\alpha_3y^3. \qquad (4.76)$$

Here F and f – "elasticities" of bonds in the molecule. Terms of the third order describe the different nonlinear effects connected with motions of atoms and electrons and with their interaction. Coefficient α determines the anharmonicity of oscillation x, coefficient α_3 – the anharmonicity of electron oscillations (see Chapter I), and coefficients α_2 and α – nonlinear interactions of oscillations x and y. It is not difficult to see that coefficient α determines the phenomenon of combination scattering well-known in optics. Actually, taking into account this term the equation of motion for the normal coordinate y in the presence of an external electrical field has the form:

$$M_s\frac{d^2y}{dt^2}+R_s\frac{dy}{dt}+fy+2\alpha xy=eE. \qquad (4.77)$$

(the molecule is considered isotropic).

With the help of equation (4.77) it is possible to construct the usual "modulation" theory of combination scattering. Here assigned oscillations

$$x = X(t)\exp i\,[\omega_0 t + \varphi(t)] + \text{ complex conjugate} \qquad (4.78)$$

(amplitude $X(t)$ and phase $\varphi(t)$. in general, are random functions of time, inasmuch as the motion x is thermal) modulate the natural frequency of the electron oscillations:

$$\omega_э = \omega_{э0}\{1 + m(t)\cos[\omega_0 t + \varphi(t)]\}, \qquad (4.79)$$

where $\omega_{э0} = \frac{f}{M_э}$. In the presence of modulation of the form (4.79) the spectrum of the field dispersed with respect to the molecule contains, obviously, besides the frequency of the incident wave ω, components at frequencies $\omega - \omega_0$ and $\omega + \omega_0$ – so-called "Stokes" and "anti-Stokes" spectral components.

It is necessary to consider, however that not always can oscillations of x be examined as assigned. Actually, taking into account the term αxy^2 the equation for the oscillation x has the form:

$$M\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Fx + \alpha y^2 = 0 \qquad (4.80)$$

(filed $\mathbf{E}$ does not directly act on oscillation x; calculation of forces of thermal origin inducing oscillations (4.78) for the future is immaterial).

If field $\mathbf{E}$ is harmonic

$$\mathbf{E} = \mathbf{E}_0\exp i(\omega t - \mathbf{kr})$$

and $\omega \gg \omega_0$, the presence of term αy^2 in (4.80) does not play an important role: it is possible not to consider influences at frequencies $\omega' = 0$ and $\omega'' = 2\omega$ connected with it. However, if field $\mathbf{E}$ is a superposition of two waves, the difference frequency of which $\omega_1 - \omega_2 \simeq \omega_0$ the term αy^2

has on oscillations an x resonance effect. In the first case the simple "modulation" treatment of combination scattering is inapplicable; and there appears a more complex effect, which we will subsequently call "forced" combination scattering.

Let us examine forced scattering in more detail. Let us assume that on falling the isotropic medium described by (4.76), (4.77), (4.80) are two monochromatic waves, which we will call the wave of the signal (frequency ω_c) and wave of pumping (frequency $\omega_н$) so that:

$$\mathbf{E} = \mathbf{E}_c + \mathbf{E}_н = \mathbf{E}_{c0} \exp i(\omega_c t - \mathbf{k}_c \mathbf{r}) + \mathbf{E}_{н0} \exp i(\omega_н t - \mathbf{k}_н \mathbf{r}). \tag{4.81}$$

We will consider that vectors $\mathbf{k}_c$ and $\mathbf{k}_н$ are collinear; let us direct the z axis along the normal to the boundary.

Amplitudes of waves in the medium, as usual, will be considered slowly changing functions z. For simplicity let us assume also that $\omega_c, \omega_н \ll \omega_{э0}$, so that instead of equation (4.77) it is possible to write the "quasi-static" equation of the form:

$$f\mathbf{y} + 2\alpha x\mathbf{y} = e\mathbf{E}. \tag{4.82}$$

Being interested only in stationary forced oscillations of the molecule, let us look for solutions for coordinates x and y in the form:

$$x = Xe^{i(\omega_н - \omega_c)t} + \text{complex conjugate};$$
$$\mathbf{y} = \mathbf{Y}_н e^{i\omega_н t} + \mathbf{Y}_c e^{i\omega_c t} + \text{complex conjugate}. \tag{4.83}$$

Here, in general, it is not assumed that

$$\omega_н - \omega_c = \omega_0 = \sqrt{\frac{F}{M}}.$$

From equations (4.80) and (4.82) there are relations:

$$f\mathbf{Y}_c + \alpha X^* \mathbf{Y}_н = e\mathbf{E}_c e^{-i\mathbf{k}_c \mathbf{r}}\ ;\ f\mathbf{Y}_н + \alpha X \mathbf{Y}_c = e\mathbf{E}_н e^{-i\mathbf{k}_н \mathbf{r}}; \tag{4.84a}$$

$$\omega_0 \delta (\Delta + i) X + \frac{\alpha}{M} (\mathbf{Y}_c^* \mathbf{Y}_н) = 0, \tag{4.84b}$$

where $\Delta = \frac{1}{\omega_0 \delta} [\omega_0^2 - (\omega_н - \omega_c)^2]$; $\delta = \frac{R}{M}$.

From (4 84) there are easily obtained relation for amplitudes $\mathbf{Y}_c$ and $\mathbf{Y}_H$, which with multiplication by eN pass into expressions for amplitudes of polarization at frequencies of the signal and pumping (we retain only components connected with nonlinear terms):

$$2\pi \mathbf{P}_c = q\,\frac{(\mathbf{E}_c \mathbf{E}_H^*)}{\Delta - i}\,\mathbf{E}_H; \quad 2\pi \mathbf{P}_H = q\,\frac{(\mathbf{E}_H \mathbf{E}_c^*)}{\Delta + i}\,\mathbf{E}_c, \tag{4.85}$$

where $q = \dfrac{2\pi N a^2 e^4}{M f^4 \omega_c \delta}$.

In general polarizations of waves $\mathbf{E}_{c0}$ and $\mathbf{E}_{H0}$, entering into the examined medium do not coincide. With the propagation of the waves in a nonlinear medium, their polarizations are changed [see (4.85)].

Here one should note one peculiarity connected with the derivation of truncated equations of the nonlinear isotropic medium. Let us remember that in the propagation of waves in a greatly anisotropic medium, as can be seen from Chapter II, the weak nonlinear polarizability of the medium cannot essentially change its polarizations $\mathbf{e}_l$. Therefore, truncated equations described the change in scalar amplitude of the wave $A(r, t)$ without a change in its polarization $\mathbf{e}$. In the examined case (isotropic medium) any direction of vectors $\mathbf{E}_c$ and $\mathbf{E}_H$ lying in planes perpendicular to vectors $\mathbf{s}_c$ and $\mathbf{s}_H$, accordingly, is the natural one. Therefore, the truncated equations describing the change in amplitudes of waves both in magnitude and in direction must be vector equations.

As was already indicated, here we will limit ourselves to the simplest but also most interesting case when vectors $\mathbf{k}_c$ and $\mathbf{k}$ are collinear. Here spectral components of the vector of nonlinear polarizability $\mathbf{P}_c$ and $\mathbf{P}_H$ are located in a plane perpendicular to vectors $\mathbf{k}_c$ and $\mathbf{k}_H$. Then the truncated equations describing the behavior of the vector amplitudes of the waves, have the form (compare § 3 of Chapter II):

$$\frac{d\mathbf{E}_c}{dz} = \beta_c (1 - i\Delta)(\mathbf{E}_H^* \mathbf{E}_c)\,\mathbf{E}_H; \quad \frac{d\mathbf{E}_H}{dz} = -\beta_H (1 + i\Delta)(\mathbf{E}_H \mathbf{E}_c^*)\,\mathbf{E}_c, \tag{4.86}$$

where $\beta_{c,H} = \dfrac{k_{c,H}\, q}{\cos \hat{\alpha}_{c,H}\, 2(1+\Delta^2)}$. From equations (4.86) it immediately follows that if $\mathbf{E}_c$ and $\mathbf{E}_H$ are perpendicular, there is no interaction of waves.[1] Let us introduce new coordinate z_1, z_2 and z_3, selecting the z_3 axis along the direction of the propagation of the waves. Vector differential equations (4.86) allow several first integrals of the form:

$$\frac{E_{ci} E_{cj}^*}{\beta_c} + \frac{E_{Hi} E_{Hj}^*}{\beta_H} = \frac{C_{ij}}{\beta_c} \text{ and } [\mathbf{E}_c \mathbf{E}_H] = \mathbf{B}. \tag{4.87}$$

Here E_{ci} and E_{Hi} – projections of vectors $\mathbf{E}_c$ and $\mathbf{E}_H$ on the z_i axis; C_{ij}, and $\mathbf{B}$ – scalar and vector constants.

The first of the relations (4.87) can be interpreted as the law of conservation of the number of quanta. For example, in the case of the linear polarization of the waves, for $i=j$ the first relation of (4.87) obtains the form:

$$N_{ci} + N_{Hi} = N_{0i}, \tag{4.88}$$

where N_i – number of quanta of corresponding frequency polarized along the z_i axis, which passes through the area element, perpendicular to the z_3 axis.

Thus, just as in the theory of nonresonant parametric amplification of equation (4.86) it is possible to examine the following separately for two conditions:

1. Conditions of the amplification in the assigned field of pumping (linear conditions). In this case the second equation of (4.86) can be disregarded. Decomposing the vector amplitude of the signal $\mathbf{E}_c$ on components parallel to the field of pumping $\mathbf{E}_{c\parallel}$ and perpendicular to $\mathbf{E}_{c\perp}$ $\left(\mathbf{E}_H \hat{\ } \mathbf{E}_{c\parallel} = 0; \mathbf{E}_{c\perp} \hat{\ } \mathbf{E}_H = \frac{\pi}{2}\right)$, instead of the first equation (4.86) we have:

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[1]This is accurate only for liquids consisting of isotropic molecules.

$$\frac{dE_{c\parallel}}{dz} = \beta_c(1 - i\Delta)|\mathbf{E}_н|^2 E_{c\parallel}; \quad \frac{dE_{c\perp}}{dz} = 0 \tag{4.89a}$$

and, consequently,

$$\mathbf{E}_c = \mathbf{E}_{c\perp}(0) + \mathbf{E}_{c\parallel}(0) e^{\beta_c |\mathbf{E}_н|^2 z} \cdot e^{-i\beta_c\Delta |\mathbf{E}_н|^2 z}. \tag{4.89b}$$

Maximum amplification takes place for $\mathbf{E}_н \parallel \mathbf{E}_c(0)$ and $\Delta = 0$. Here for the factor of accretion we have $\Gamma_0 = |\mathbf{E}_н|^2 \frac{k_c q}{\cos \widehat{sz}}$. In the medium possessing losses at frequency ω_c, the exponential growth $\mathbf{E}_c$ will take place only when $\mathbf{E}_н > \mathbf{E}_{nop}$ [see (4.7)].

2. Conditions of amplification in which there becomes essential a reverse reaction of the wave of the signal on the wave of pumping are conditions of saturation of the amplifier. An analysis of these conditions represents the primary interest from the point of view of the theory of the parametric generator, which uses the phenomenon of forced combination scattering. Here equations (4.86) must be solved jointly. An analysis of equations (4.86) shows that if $\Delta \neq 0$ and vectors $\mathbf{E}_c$ and $\mathbf{E}_н$ are not collinear, the linearly polarized light passes in the examined medium into an elliptically polarized light. Here rotations of vectors $\mathbf{E}_н$ and $\mathbf{E}_c$ with the propagation of waves along the z axis occur in various directions. Conversely, when $\Delta = 0$, the linearly polarized light entering into the medium remains such even when vectors $\mathbf{E}_c$ and $\mathbf{E}_н$ are not parallel.

For linearly polarized waves, the value of angle ψ between vectors $\mathbf{E}_c$ and $\mathbf{E}_н$ is determined by the expression:

$$\sin^2 \psi = \frac{|\mathbf{B}|^2}{|\mathbf{E}_c|^2 \cdot |\mathbf{E}_н|^2}.$$

The law of the change in moduli of amplitudes $\mathbf{E}_c$ and $\mathbf{E}_н$ can be obtained if one were to use the relation

$$|(\mathbf{E}_c \mathbf{E}_н^*)|^2 + |[\mathbf{E}_c \mathbf{E}_н]|^2 = |\mathbf{E}_c|^2 \cdot |\mathbf{E}_н|^2. \tag{4.90}$$

then

$$\frac{d}{dz}|\mathbf{E}_c|^2 = 2\beta_c(|\mathbf{E}_H|^2\mathbf{E}_c|^2-|\mathbf{B}|^2);\ \frac{d}{dz}|\mathbf{E}_H|^2 = -2\beta_H(|\mathbf{E}_H|^2\cdot|\mathbf{E}_c|^2-|\mathbf{B}|^2). \qquad (4.91)$$

The solution to these equations has the form:

$$|\mathbf{E}_c|^2 = C+\Lambda\frac{1-Ge^{-2\beta_H\Lambda z}}{1+Ge^{-2\beta_H\Lambda z}};\quad |\mathbf{E}_H|^2 = \frac{\beta_H}{\beta_c}(C-|\mathbf{E}_c|^2). \qquad (4.92)$$

Here

$$\Lambda = \sqrt{C^2-\frac{\beta_c}{\beta_H}|\mathbf{B}|^2};\ 2C = C_{11}+C_{22};\quad G=\frac{\Lambda+\frac{\beta_c}{\beta_H}|\mathbf{E}_{H0}|^2}{\Lambda-\frac{\beta_c}{\beta_H}|\mathbf{E}_{H0}|^2}.$$

The solutions of (4.92), if they are examined for all values z (and not only for the positive), determine the energy transitions from the state at $z=-\infty$ when $|\mathbf{E}_c|^2 = C-\Lambda$ $|\mathbf{E}_H|^2 = \frac{\beta_H}{\beta_c}(C+\Lambda)$ and to the state at $z=+\infty$ when

$$|\mathbf{E}_c|^2 = C+\Lambda;\quad |\mathbf{E}_H|^2 = \frac{\beta_H}{\beta_c}(C-\Lambda).$$

Thus, in the process of the propagation of waves, when $\mathbf{E}_H, \mathbf{E}_c \neq 0$ the energy of the wave of pumping passes to the wave of the signal.

The maximum value of the amplitude of the signal at the outlet of the system is determined by relationship of the Manley-Rowe type [see §§ 3.4 of Chapter II):

$$E_c^2(\infty) = \frac{\omega_c}{\omega_H}E_{H0}^2. \qquad (4.93)$$

Neither parameters of the substance nor detuning $\omega_H-\omega_c)-\omega_0$ enter into (4.93). Of course the less distance at which there is attained a maximum power of the signal, the less the detuning and the greater the parameter q.

If the examined medium is placed in the Fabry-Perot resonator, tuned to a frequency $\sim\omega_H-\omega_0$, in it parametric oscillations at a

frequency equal to $\omega_H - \omega_0$ can be self-excited. The condition of self-excitation of parametric generator can be obtained absolutely analogous to that which was done in § 3 of this chapter; here one should consider that in the parametric generator, using forced combination scattering, amplification of the signal takes place both for direct and for return waves.

Designating by R_1 and R_2 reflection factors of mirrors at frequency $\omega_H - \omega_0$, for the condition of self-excitation we have:

$$\frac{2\beta_c E_{H0}^2 \cos^2\psi}{k_c} > \frac{1 - R_1 R_2}{k_c d} = \frac{1}{Q} \qquad (4.94)$$

d – distance between the mirrors (compare (4.47)).

The magnitude of the stationary amplitude in the examined generator is determined by the reaction of parametrically excited oscillations on pumping; therefore, formula (4.93) together with the maximum amplitude of the signal at the outlet of the amplifier determines in order of magnitude the efficiency of the generator. Let us note that inasmuch as usually $\frac{\omega_H}{\omega_0} \gg 1$ (frequency of the signal lies in the optical range, and frequency of symmetric oscillations – in the infrared) the efficiency of the parametric generator, which uses the phenomenon of forced combination scattering, should be quite high and reach tens of percent. The polarization of oscillations of the generator will coincide with the polarization of the wave of pumping.

It is interesting to compare characteristics of amplifiers and generators using the phenomenon of forced combination scattering with characteristics of similar devices using the nonlinearity of electron polarizability (see §§ 2-4 of this chapter). In both cases the process of amplification is the result of the disintegration of photons of pumping, however, if for nonresonant interactions with disintegration of a photon of frequency ω_H there appear two photons of frequencies ω_1, ω_2 $(\omega_1 + \omega_2 = \omega_H)$, with forced combination scattering part of the energy of the photon ω_H is transmitted to the wave of

frequency $\omega_н - \omega_0$, and the remainder is transmitted to molecular oscillations at frequency ω_0. The last circumstance is the reason for the fact that with forced combination scattering the stored interactions occur independently of dispersion properties of the medium. The energy exchange between waves of pumping and the signal is accomplished by the means of molecular oscillations; the phase of the latter is established each time as optimum from the point of view of energy transfer from the wave of pumping to the wave of the signal. With nonresonant interactions the indicated energy exchange is carried out with the help of the electromagnetic wave of the difference frequency; its phase is determined by dispersion properties of the medium. In accordance with what has been said, if conditions of synchronism for waves of frequencies $\omega_н, \omega_c$ and $\omega_н - \omega_c$ cannot be carried out, the appearance even small dipole moments for molecular oscillations (weak coupling) can considerably worsen characteristics of amplifiers and generators on forced combination scattering.

The band of the amplifier on forced combination scattering is determined by the quantity δ, i.e., relaxation time of oscillation x.

Thus far we were limited to examination of behavior of only Stokes components of lines of combination scattering in the field of the intense wave of pumping.

Interesting effects, in a certain sense similar to those examined in 2.3 of § 2 of this chapter to parametric effects with low-frequency pumping, can be observed on anti-Stokes components. In this case the field in the medium should be presented in the form of the superposition of not two but at least three waves: with frequencies $\omega_н$; $\omega_c \simeq \omega_н - \omega_0$; $\omega_a \simeq \omega_н + \omega_0$ [compare with (4.21)]

$$\begin{aligned}\mathbf{E} = \mathbf{E}_н + \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_н(\mu z)\exp i(\omega_н t - \mathbf{k}_н\mathbf{r}) + \\ + \mathbf{E}_c(\mu z)\exp i(\omega_1 t - \mathbf{k}_1\mathbf{r}) + \mathbf{E}_a(\mu z)\exp i(\omega_2 t - \mathbf{k}_2\mathbf{r}).\end{aligned} \quad (4\ 95)$$

For vector amplitudes $\mathbf{E}_н$, $\mathbf{E}_c$, and $\mathbf{E}_a$ there can be obtained truncated equations absolutely similar to that which was done above.

Conducting the corresponding computations, we arrive in the examined case to the system of three truncated equations for amplitudes $\mathbf{E}_н$, $\mathbf{E}_c$, and $\mathbf{E}_a$. In the assigned field of pumping the system is reduced to two equations for slowly changing amplitudes $\mathbf{E}_c$ and $\mathbf{E}_a$:

$$\frac{d\mathbf{E}_c}{dz}=\beta_c(1-i\Delta)\left[(\mathbf{E}_a^*\mathbf{E}_н)e^{i(\mathbf{k}_c+\mathbf{k}_a-2\mathbf{k}_н)r}+(\mathbf{E}_н^*\mathbf{E}_c)\right]\mathbf{E}_н; \qquad (4.96)$$

$$\frac{d\mathbf{E}_a}{dz}=-\beta_a(1+i\Delta)\left[(\mathbf{E}_a\mathbf{E}_н^*)+(\mathbf{E}_н\mathbf{E}_c^*)e^{i(\mathbf{k}_c+\mathbf{k}_a-2\mathbf{k}_н)r}\right]\mathbf{E}_н. \qquad (4.97)$$

Right sides of equations (4.96) and (4.97) do not contain oscillatory terms and, consequently, stored effects are possible if

$$2\mathbf{k}_н\simeq\mathbf{k}_c+\mathbf{k}_a. \qquad (4.98)$$

Thus, if the stored interaction of the field of pumping with the Stokes component takes place in a wide interval of angles $\mathbf{k}_н\hat{\ }\mathbf{k}_c$ (let us note that collinear vectors $\mathbf{k}_н$ and $\mathbf{k}_c$ were introduced above only for simplicity), the stored interaction of the field of pumping the anti-Stokes component takes place only in fixed directions determined by formula (4.98).

Physical meaning of (4.98) can be explained in the following way. The energy exchange between waves with frequencies $\omega_н$ and ω_c and $\omega_н$ and ω_a is produced by the means of the same molecular oscillations having the frequency ω_0. Both shown interaction will lead to stored effects, if optimum energy exchange for them was carried out during the same phase of molecular oscillations (compare (4.18)).

An analysis of equations (4.96) and (4.97) will be conducted in the simplest case $\Delta=0$; the amplitude of pumping will be considered real, and for simplicity we disregard dispersion of the medium in band $\omega_н-\omega_0$, $\omega_н+\omega_0$. Then equations (4.96) and (4.97) become scalar (condition (4.98) is fulfilled for one-dimensional interaction) and acquire the form:

$$\frac{dE_c}{dz}=\Gamma_c(E_c+E_a); \qquad (4.99)$$

$$\frac{dE_a}{dz}=-\Gamma_a(E_c+E_a), \qquad (4.100)$$

where $\Gamma_c = \beta_c E_H^2;\ \Gamma_a = \beta_a E_H^2.$

It is easy to obtain a general solution of these equations for conditions:

$$z=0;\ E_c(0)=E_{co};\ E_a(0)=E_{ao}$$

It has the form:

$$E_a(z) = -\frac{1}{\Gamma_a-\Gamma_c}\left\{\Gamma_a E_{co}\left[1-e^{-(\Gamma_a-\Gamma_c)z}\right]+\right.$$
$$\left.+E_{ao}\left[\Gamma_c-\Gamma_a e^{-(\Gamma_a-\Gamma_c)z}\right]\right\} \qquad (4.101)$$
$$E_c(z) = \frac{1}{\Gamma_a-\Gamma_c}\left\{\Gamma_c E_{ao}\left[1-e^{-(\Gamma_a-\Gamma_c)z}\right]+\right.$$
$$\left.+E_{co}\left[\Gamma_a-\Gamma_c e^{-(\Gamma_a-\Gamma_c)z}\right]\right\}. \qquad (4.102)$$

From expressions (4.101)-(4.102) it follows that when $z\to\infty$ amplitudes $E_c(z)$ and $E_a(z)$ approach stabilized values determined by relationships:

$$E_{cy} = \frac{1}{\Gamma_a-\Gamma_c}\left[\Gamma_c E_{ao}+\Gamma_a E_{co}\right]; \qquad (4.103)$$
$$E_{ay} = -\frac{1}{\Gamma_a-\Gamma_c}\left[\Gamma_a E_{co}+\Gamma_c E_{ao}\right]. \qquad (4.104)$$

The character of the change in amplitudes $E_c(z)$ and $E_a(z)$ with the coordinate is determined by the relationship of the boundary amplitudes. If

$$E_{ao}+E_{co}>0, \qquad (4.105)$$

then $-E_{ay}>-E_{ao}$, and amplification of the anti-Stokes component takes place. With fulfillment of the inequality opposite the inequality (4.105), $-E_{ay}<-E_{ao}$ and the amplitude of the anti-Stoke component decreases with distance.

It is necessary to note that in an isotropic dispersive medium condition (4.98) cannot be carried out for waves of one direction; therefore, coherent radiation of anti-Stoke components in a liquid excited by an intense parallel beam of the laser, occurs in the cone by the solution $\theta^2 \simeq \frac{1}{n_H}\frac{\omega_c}{\omega_a}(n_a-n_c)$, the axis of which coincides with the

axis of the beam of basic radiation. Here n_H, n_a, and n_c are indices of refraction. The corresponding radiation of Stokes components in this case also occurs at an angle to the vector $\mathbf{k}_H$. Finally, besides components $\omega_H - \omega_0$ and $\omega_H + \omega_0$, in the medium highest combination frequencies are also excited – $\omega_H \pm n\omega_0$ ($n=2, 3...$, see also [205, 208]). The spatial structure of this radiation (see Fig. 4-11) can be established on the basis of an analysis of dispersion relationships of the type (4.98).

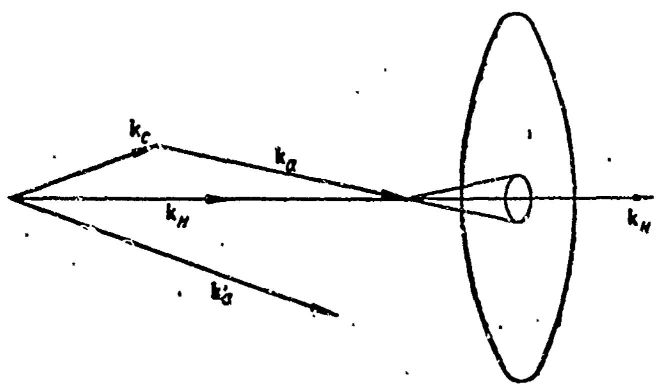

Fig. 4-11. Diagram characterizing the spatial structure of the radiation of anti-Stokes components with forced combination scattering of a plane monochromatic wave in an isotropic dispersive medium. The anti-Stokes component with a frequency $\omega_a = \omega_H + \omega_0$ is radiated in the cone of directions determined by the relation $2\mathbf{k}_H = \mathbf{k}_c + \mathbf{k}_a$. Here directions are shown in which there occurs radiation of the component of a higher order $\omega_a' = \omega_H + 2\omega_0$ $\mathbf{k}_a' = 3\mathbf{k}_H - 2\mathbf{k}_c$.

Thus, the expounded theory of forced combination scattering permits not only giving a qualitative treatment of the mechanism of the phenomenon but also obtaining a number of quantitative results. Here constants (α, β) should be determined from the quantum-mechanical calculation or by experimental means. Quantum treatment of the phenomenon of forced combination scattering is given in [156] and [200]. It is necessary to note, however, that in the mentioned works the analysis is limited only to the outlet of conditions of the excitation of oscillations on Stokes components (for parametric

effects in quantum systems see also (163)). In conclusion of this section one should stress that the problem above examined on forced combination scattering is the simplest. Let us note, first of all, that calculation of the anisotropy of electrical properties of molecules leads to the conclusion concerning the possibility of interaction in the medium consisting of such molecules of waves at frequencies $\omega_н$ and $\omega_с \simeq \omega_н - \omega_0$ (Rayleigh and Stokes component) with any polarizations of the indicated waves (compare with equations (4.86)).

Interesting parametric effects can be connected with terms of the fourth order in the decomposition of potential energy (4.76). A diagram of the classical calculation of the indicated effects — effects of forced combination scattering of the second order,[1] is analogous to that stated above. An addition to the potential energy (4.76) in the simplest case of two normal oscillations has the form:

$$\Delta U = \beta_1 x^4 + \beta_2 x^3 y + \beta_3 x^2 y^2 + \beta_4 x y^3 + \beta_5 y^4. \qquad (4.106)$$

The passive combination scattering of the second order is described by the term with β_3; in equation (4.77) force of the form $2\beta_3 x^2 y$ is connected with it. A reverse reaction to the molecular oscillations is carried out due to the force $2\beta_3 x y^2$ in equation (4.80). The latter means that in contrast to scattering of the first order a reverse reaction here has the character of the parametric effect on oscillations x (see 4.80). Coherent molecular oscillations here can appear only under the condition of an excess in the threshold of parametric excitation. An interesting effect can be connected with the term at β_4: here appearance of forced combination scattering with a frequency $\omega_н' + \omega_н'' - \omega_0$ is possible with excitation of the medium by biharmonic pumping of the form:

$$\mathbf{E}_н = \mathbf{E}_н' \exp i\,(\omega_н' t - \mathbf{k}_н' \mathbf{r}) + \mathbf{E}_н'' \exp i\,(\omega_н'' t - \mathbf{k}_н'' \mathbf{r}) \qquad (4.107)$$

(in the degenerated case $\omega_н' = \omega_н'' = \omega_н$ the "Stokes" component $2\omega_н - \omega_0$ appears).

[1]For the usual, "passive," scattering of the second order, see Ye. F. Gross, P. Pavinskiy, A. Stekhanov UFN, 1951, XLIII, No. 4, 536.

For an interaction of the last type coherent molecular oscillations are excited just as in the case of scattering of the first order (corresponding force in (4.80) $\sim\beta_4 y^3$). Let us note that the threshold of forced scattering of the second order for both examined variants is very high.

C H A P T E R   V

MODULATED WAVES IN NONLINEAR DISPERSIVE MEDIA

§ 1. Introduction

Thus far, in examining nonlinear wave interactions in dispersive media we were limited to cases when amplitudes and phases of the interacting waves do not depend on time (waves are unmodulated). At the same time, problems on nonlinear interactions of modulated waves now play a very important role in nonlinear optics. In this region it is possible to distinguish two classes of problems:

1. Problems connected with the investigation of the process of modulation of light waves in nonlinear media (see, for example, [165]-[172]).

2. Problems, connected with the investigation of distortions of the form of modulation, with propagation of the modulated wave in a nonlinear medium or a medium with variable parameters (see, for example, [56], [173]).

For description of regularities of the propagation of modulated waves in a weakly nonlinear dispersive medium, the method of slowly changing amplitudes can be used (see Chapter II). Inasmuch as the complex amplitudes are changed in this case both in space and time, truncated equations acquire the form of partial differential equations. The solution of them becomes, in most cases, complex and can be conducted only by means of numerical integration. Only for the simplest problems can there be obtained an explicit solution, and on

its basis the process of the propagation of light waves is analyzed. Two such problems are expounded in this chapter. The first problem is the modulation of a light wave with passage of it through the electro-optical medium, which is found in the low-frequency (as compared to optical frequencies) electromagnetic field, and the second problem is the passage of a modulated wave through the parametric amplifier. Both these problems are analyzed in a parametric approximation, i.e., in the approximation when the field of one of the waves can be considered assigned. In Chapter IV it was shown that in the parametric amplifier, while wave of signal did not grow in amplitude up to the value comparable with the amplitude of the wave of pumping, such an approximation is valid. An absolutely analogous position takes place in the case of the propagation of a light wave on a nonlinear medium occurring in a low-frequency electromagnetic field. In this case the action from the side of the light wave on the low-frequency field can be disregarded, and only the influence in the low-frequency field on the field of the light wave can be examined.

For the foundation of the possibility of examining the behavior of the light wave in a parametric approximation, let us turn to the interaction of three waves, studied in Chapters II, IV in a medium with nonlinearity of the quadratic type and analyze the case when the frequency of one of the waves is considerably lower than frequencies of the other two. With incidence on the boundary of nonlinear and linear media of two waves with frequencies ω_1 and ω_2 (in this case $\omega_1 \ll \omega_2$) and comparable amplitudes, in the nonlinear medium there appear waves of sum and difference frequencies:

$$\omega_3 = \omega_2 \pm \omega_1. \tag{5.1}$$

If the wave vector of the appearing wave satisfies the condition of synchronism:

$$\mathbf{k}_3 = \mathbf{k}_2 \pm \mathbf{k}_1, \tag{5.2}$$

the amplitude of the corresponding wave grows with distance until the amplitude of the wave with index "2" falls to zero. With

fulfillment of the condition of synchronism for the sum frequency ("+" sign in (5.2)) the amplitude of the wave with subscript "1" decreases down to a certain value A_{1min}. If, however, the condition of synchronism is fulfilled for the difference frequency ("-" sign in (5.2)), the amplitude of the wave with subscript "1" increases down to value A_{1max}. In the case $\omega_1 \ll \omega_2$ the drop $A_{10}^2 - A_{1min}^2$ or $A_{1max}^2 - A_{10}^2$ is equal to (see § 4, Chapter II):

$$A_{10}^2 - A_{1min}^2 = A_{1max}^2 - A_{10}^2 = \frac{\omega_1}{\omega_2} A_{10}^2. \tag{5.3}$$

Thus, the amplitude of the low-frequency wave practically does not change which gives the basis to disregard the effect on this wave from the direction of light waves. This means that the process of the interaction of traveling waves in the fulfillment of condition $\omega_1 \ll \omega_2$ can be described quite accurately in the parametric approximation. Similarly, such an approximation is admissible in the case of the general form of the low-frequency field.

The process of the modulation of light occurs differently in anisotropic and isotropic media. In anisotropic nonlinear media (for example, in KDP and ADP crystals) phase modulation of linearly polarized waves is carried out. In isotropic media (for example, crystals CuCl, ZnS and others) and also in anisotropic media in directions of isotropy with modulation elliptically polarized light will be formed. Therefore, an examination of the process of modulation in anisotropic and isotropic media will be conducted separately.

§ 2. Modulation of Light in Optically Anisotropic Crystals

If a light wave propagates in a quadratic medium occurring under the effect of a modulating electrical field $\mathbf{E}_m(\mathbf{r}, t)$, then the complex amplitude of this wave $A(\mathbf{r}, t)$, as follows from Chapter II, is described by a truncated equation of the form:

$$[\mathbf{e}[\mathbf{k}\mathbf{e}]]\, s\frac{\partial A}{\partial t} + [\mathbf{e}[\mathbf{k}\mathbf{e}]]\nabla A = -\frac{2\pi i\omega^2}{c^2}(\mathbf{e}\hat{\chi}\mathbf{e}\mathbf{E}_m)A. \tag{5.4}$$

Here, just as in the second chapter, $\mathbf{e}$ – unit vector directed over the electric field strength of the light wave, $\mathbf{k}$ and $\mathbf{s}$ – its wave and beam vectors and $\hat{\chi}$ – operator of the quadratic nonlinear polarization. As was already underlined in the second chapter, the equation of the type (5.4) determines the change in amplitude A along the direction of beam vector $\mathbf{s}$. In a direction perpendicular to $\mathbf{s}$, equation (5.4) does not describe the change in amplitude, and it is determined only by specific subsidiary conditions of the problem. These conditions include properties of the medium, boundary conditions, and the form of the modulating field. Considering that $A=A_0e^{i\varphi}$, we will obtain for ∇A the expression:

$$\nabla A=(\nabla A_0+iA_0\nabla\varphi)e^{i\varphi}. \qquad (5.5)$$

In this expression directions ∇A_0 and $\nabla\varphi$ in general are different. If the crystal occupies the half-space $x'>0$, then ∇A_0 is directed perpendicular to the plane of division of the media, i.e., along the x' axis. If the field $\mathbf{E}_m$ constitutes a plane traveling or standing wave, then the direction $\nabla\varphi$ coincides with direction $\mathbf{k}_m$ of the wave vector of this wave.

Let us consider at first the case when the field of modulation $\mathbf{E}_m$ has the form:

$$\mathbf{E}_m=\mathbf{E}_m^0\cos(\omega_m t-\mathbf{k}_m\mathbf{r}), \qquad (5.6)$$

Then truncated equations for amplitude A_0 and ϕ will be recorded in the following form:

$$\begin{gathered}\frac{\partial A_0}{\partial x'}+\frac{1}{v_{rp}\cos\widehat{sx'}}\cdot\frac{\partial A_0}{\partial t}=0;\\ \frac{\partial\varphi}{\partial\xi}+\frac{1}{v_{rp}\cdot\cos\widehat{sk_m}}\frac{\partial\varphi}{\partial t}=-B\cos(\omega_m t-k_m\xi),\end{gathered} \qquad (5.7)$$

where ξ – coordinate in the direction of the vector $\mathbf{k}_m$, v_{rp} – group speed, and

$$B=\frac{2\pi\omega^2}{kc^2}\cdot\frac{\left(\mathbf{e}\hat{\chi}\mathbf{e}\mathbf{E}_m^0\right)}{\cos\widehat{sk}\cdot\cos\widehat{sk_m}}. \qquad (5.8)$$

From equations (5.7) one can see, first of all, that with incidence on the crystal of unmodulated light, in the passing wave there appears purely phase modulation. Integrating the second of these equations, we have:

$$\varphi = \varphi_0 - B\xi \frac{\sin \Delta^{(-)}\xi}{\Delta^{(-)}\xi} \cos(\omega_m t - \mathbf{k}_m \mathbf{r} - \Delta^{(-)}\xi), \tag{5.9}$$

where

$$\Delta^{(-)} = \frac{\omega_m - \mathbf{k}_m v_{rp} \cos \widehat{\mathbf{s}\mathbf{k}_m}}{2 v_{rp} \cos \widehat{\mathbf{s}\mathbf{k}_m}}. \tag{5.10}$$

Expression (5.9) characterizes modulation of the phase in the examined case.

With fulfillment of the relation

$$\frac{\omega_m}{k_m} = v_{rp} \cos \widehat{\mathbf{s}\mathbf{k}_m} \tag{5.11}$$

the index of modulation for given ξ is maximum and grows linearly with distance along the direction of propagation of the modulating wave. Here, as one can see from (5.11), the component of the group velocity of light on the direction of propagation of the modulating wave is equal to the phase speed of this wave. Condition (5.11) is the condition of synchronism of the wave of modulation and all spectral components of the light wave. Actually, for the spectral component of frequency $\omega^+ = \omega + \omega_m$, the condition of synchronism has the form:

$$\mathbf{k}^+ = \mathbf{k} + \mathbf{k}_m. \tag{5.12}$$

Expanding function $\omega(\mathbf{k}^+) = \omega(\mathbf{k} + \mathbf{k}_m)$ in series and considering that $\mathbf{k}_m$ in absolute value is many orders less than $\mathbf{k}$, we can obtain the relation:

$$\frac{\partial \omega}{\partial \mathbf{k}} \mathbf{k}_m = \omega_m, \tag{5.13}$$

which coincides with (5.11).

Thus, for amplitude $\mathbf{E}_0$ of the electrical field of modulated light wave with fulfillment of the condition of synchronism we have:

$$\mathbf{E}_0 = \mathbf{e}A_0 e^{i[\varphi_0 - B\xi\cos(\omega_m t - \mathbf{k}_m \mathbf{r})]} \quad (5.14)$$

If one were to set the defined value ξ, then in this section the index of modulation $m_\Phi = B\xi$ will be constant. As is known from the theory of phase modulation, amplitudes of combination components of a wave with frequencies $\omega \pm n\omega_m$ are determined by Bessel functions $J_n(m_\Phi)$. With a change in ξ the spectrum of the light wave is transformed. When $\sim m_\Phi = 2{,}4$, for example, the amplitude of the component of frequency ω turns into zero, i.e., the energy of the wave completely turns into side frequencies. At fixed ξ tuning can be produced by a change in amplitude of the modulating wave. Experimental realization of such a scheme of modulation is the subject of work [169].

Let us examine now the case when the modulating field $\mathbf{E}_m$ has the form of the standing wave

$$\mathbf{E}_m = \mathbf{E}_m^0 \cos\omega_m t \cdot \cos \mathbf{k}_m \mathbf{r}. \quad (5.15)$$

By presenting the standing wave in the form of the superposition of two traveling waves, it is possible to obtain immediately the solution of the truncated equation for the phase in the form:

$$\varphi = \varphi_0 - \frac{1}{2} B\xi \frac{\sin \Delta^{(-)}\xi}{\Delta^{(-)}\xi} \cos(\omega_m t - \mathbf{k}_m \mathbf{r} - \Delta^{(-)}\xi) - \\ - \frac{1}{2} B\xi \frac{\sin \Delta^{(+)}\xi}{\Delta^{(+)}\xi} \cos(\omega_m t + \mathbf{k}_m \mathbf{r} - \Delta^{(+)}\xi), \quad (5.16)$$

where $\Delta^{(-)}$ is determined by expression (5.10), and

$$\Delta^{(+)} = \frac{\omega_m + k_m v_{\text{гр}} \cos \widehat{s k_m}}{2 v_{\text{гр}} \cos \widehat{s k_m}}. \quad (5.17)$$

At first glance it is natural to strive to ensure the synchronism between the light wave and one of the traveling waves of low

frequency, for example, the first in expression (5.16). The second wave, traveling in the opposite direction, interacts with the light wave considerably more weakly and does not give an accumulation effect. However, the given reasoning has meaning only for those cases several half-waves of the modulating frequency fit on the length of the resonator. If, however, on the length of the resonator there is less than half of a wavelength, then each of the traveling waves of modulation in equal degree interacts with the light wave, and fulfillment of the condition of synchronism is not obligatory. Indeed, let us assume that synchronism is ensured for the first wave in (5.16), i.e., $\Delta^{(-)}=0$. Then $\Delta^{+}=k_m$ and

$$\varphi=\varphi_0-\frac{1}{2}B\xi\cos(\omega_m t-k_m\xi)-\frac{1}{2}B\xi\frac{\sin k_m\xi}{k_m\xi}\cos\omega_m t. \qquad (5.18)$$

If one were to designate the index of modulation by the first wave $m_{\phi 1}$, and the second $m_{\phi 2}$, then the quantity

$$\frac{m_{\phi 2}}{m_{\phi 1}}=\frac{\sin k_m\xi}{k_m\xi} \qquad (5.19)$$

characterizes the relative contribution of waves into the modulation of light wave. When $k_m\xi\ll 1$ quantity $\frac{m_{\phi 2}}{m_{\phi 1}}\simeq 1$. With a growth in ξ this ratio decreases, since the accumulation effect appears. Such a modulator is described in [167].

The field $\mathbf{E}_m$ uniform in space is a special case of a plane standing wave, when $k_m=0$. We have then $\Delta^{(+)}=\Delta^{(-)}=\frac{\omega_m}{2v_{\text{гр}}}=\Delta$ and

$$\varphi=\varphi_0-B\frac{\sin\Delta\xi}{\Delta}\cos(\omega_m t-\Delta\xi), \qquad (5.20)$$

where ξ – coordinate along the direction of the ray of light. (A corresponding experiment is described in [193]).

§ 3. Devices with Prolonged Effect of the Field of Modulation on a Light Wave

Modulators of light which use the effects of synchronism of the light wave and wave modulation, have a number of important advantages

as compared to modulators of other types. One of them, moderate power, is consumable to modulation. Let us consider, therefore, certain questions referring to modulators with one traveling wave of modulation occurring in synchronism with the light wave. If the counter wave of modulation is present which in a number of devices of such type takes place, then its action is little, and in the first approximation it can be disregarded.

Above, in examining the modulation of light properties of the medium were described by general tensor $\hat{\chi}$, which in the transition to the definite type of crystal is specified. Let us clarify the possibility of light modulation with the help of crystals of dihydrophosphates of potassium and ammonium (**KDP** and **ADP**), which in an optical respect are uniaxial. According to [194] the linear electro-optical effect is greatly expressed in these crystals only when the field is applied along the optical axis. Consequently, with realization of modulators of light on crystals **KDP** and **ADP**, the modulating electrical field should be directed along the optical axis (z axis).

Let us now consider the question on the polarization of light waves. Wave vector $\mathbf{k}_m$ is several orders less than vector $\mathbf{k}$. Therefore, for fulfillment of conditions of synchronism of the type (5.12), wave vectors of the initial light wave and lateral components appearing in the medium should be close in the direction and in magnitude. On the one hand, even for a modulating frequency of 10 G-Hz the distinction of them in magnitude should appear only in the fourth sign. On the other hand, indices of refraction for ordinary and extraordinary waves already differ in the second sign. Consequently, all spectral components must have one polarization – be either ordinary or extraordinary. A light wave with an arbitrary direction of propagation and arbitrary polarization due to double refraction breaks up into an ordinary and extraordinary wave. In virtue of the considerable distinction of indices of refraction for these waves, the interaction between them is impossible. Each of the waves with propagation is subjected, in general, to modulation.

The modulation of light is directly connected with properties of symmetry of the crystals **KDP** and **ADP**. These properties of

symmetry are such that in the expression for the nonlinear part of polarization:

$$P_i = \chi_{ijk} E_j E_k \tag{5.21}$$

all subscripts i, i and k must be different (see Chapter I, § 7). Inasmuch as along the optical axis (z axis) there is directed the field $\mathbf{E}_m$, the light wave should have components on other axes of the crystal (let us designate them x' and y'). Only under this condition does polarization at side frequencies appear, and it is always located in the plane $x'y'$. Waves of side frequencies appear only in the case when the vector of nonlinear polarization has a component on the electrical field of side frequency.

Let us consider as an example the propagation of a ray of light perpendicular to the optical axis (Fig. 5.1). For the most effective modulation, in this case it is necessary that vector $\mathbf{E}_0$ be oriented in one of the directions [1, 1.0], [1, -1.0]. Only in these cases does the nonlinear part of polarization, determined by relationship [5.21], coincide in the direction with vector $\mathbf{E}_0$. Coefficient B, determined by the relation (5.8), is equal in the examined case to

$$B = \frac{\omega^2}{2c^2} \cdot \frac{\Delta\varepsilon}{k \cos \widehat{\mathbf{k}\mathbf{k}_{ox}}}, \tag{5.22}$$

where $\Delta\varepsilon$ – change in dielectric constant of the crystal under the action of the field $E_m - \Delta\varepsilon = 4\pi\chi E_m^0$. Let us find this value. Prior to superposition of the field, section $z=0$ of the ellipsoid of indices of refraction was circular and the dielectric constant equaled ε_0. After superposition of the field E_m^0 along the z axis the circumference of the section is turned into an ellipse, the principal axes of which pass at an angle of 45° to axes x' and y' of the crystal. According to [194] corresponding values ε along the principal axes are determined by equalities:

$$\frac{1}{\varepsilon_1} = \frac{1}{\varepsilon_0} - r_{63} E_m^0; \quad \frac{1}{\varepsilon_2} = \frac{1}{\varepsilon_0} + r_{63} E_m^0, \tag{5.23}$$

where r_{63} – electro-optical coefficient, which consists of, for

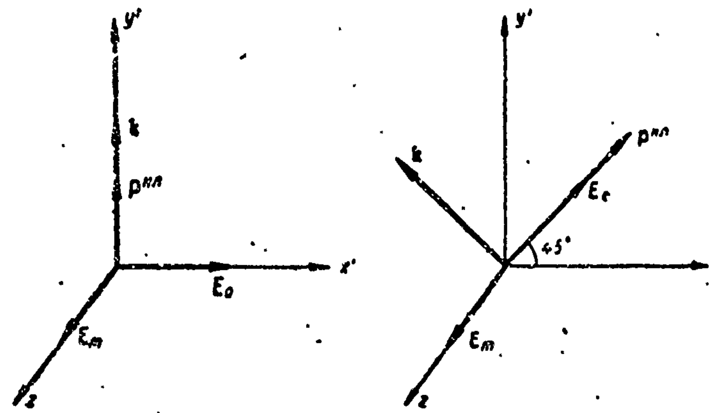

Fig. 5.1. Concerning the question of direction **P** – nonlinear part of polarization appearing from $\mathbf{E}_0$ and $\mathbf{E}_m$, and "feeding" wave of side frequencies: x', y', z – axes of the crystal (scale is not maintained).

example, for KDP $8{,}47\cdot 10^{-10}$ cm/V [169]. Hence, in virtue of the smallness of modulation factors

$$\varepsilon_1=\varepsilon_0\left(1+\varepsilon_0 r_{63}E_m^0\right);\quad \varepsilon_2=\varepsilon_0\left(1-\varepsilon_0 r_{63}E_m^0\right) \tag{5.24}$$

and

$$\Delta\varepsilon=\varepsilon_0^2 r_{63}E_m^0. \tag{5.25}$$

In case of synchronism the modulation factor of phase m_ϕ is equal to

$$m_\phi=Bx\simeq 3\pi\frac{n_0^3 r_{63}E_m^0 x}{\lambda}, \tag{5.26}$$

where it is considered that for KDP crystals with synchronism $\cos\widehat{\mathbf{k}\mathbf{k}_m}\simeq\frac{1}{3}$ (see [166]). From formula (5.26) it follows that when $\lambda=7000\,\mathrm{\mathring{A}}$, $n_0=1{,}5$ and $E_m^0=50$ V/cm the modulation factor of phase $m_\phi=1$ is attained at distance $x\simeq 50$ cm.

§ 4. Modulation of Light in Optically Isotropic Crystals[1]

Let us now turn to examination of the process of modulation of a light wave with passage through nonlinear crystals of cubic structure (classes T and T_d). It is obvious that such crystals are, in absence of a modulating field, isotropic. Therefore, the process of the propagation of light through a crystal, to which a modulating field is applied, is described in a parametric approximation by the equation:[2]

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}^{(s)}}{\partial t^2} = \frac{4\pi}{c^2}\hat{\chi}\mathbf{E}_m\frac{\partial^2 \mathbf{E}}{\partial t^2}; \qquad (5.27)$$

[1]Interest toward isotropic crystals as light modulators is explained by the following. In the use of anisotropic crystals for phase light modulation with rotation of the plane of polarization, which after passage by the light wave of a Nicol prism is turned into amplitude modulation, the ray of light should be directed along the optical axis. In the same direction there should be applied a modulating electrical field, which creates considerable design difficulties. Furthermore, the ray of light should be parallel to the optical axis, which puts limitation on the divergence of the light ray.

The difficulties indicated above do not exist in the case of the use of isotropic crystals as modulators of light. Here there are no such serious limitations on the parallelism of the ray of light. In [186] there is discussion about the satisfactory modulation of light with divergence of the ray up to 20°. In isotropic crystals the modulating field can be applied in a direction perpendicular to the direction of the propagation of light. Here it is possible to carry out modulation immediately by two signals applied in mutually-perpendicular directions [224]. Regarding, however, the magnitude of the electro-optical coefficient, then, for example, for ZnS it is only 4 times less than that for KDP, and consists of according to data given in [224] and [186], $r_{41} \simeq 2 \cdot 10^{-10}$ cm/V.

[2]In the given equation instead of rot rot $\mathbf{E}$ there is recorded $-\nabla^2\mathbf{E}$. This is connected with the fact that at the force standing in the right side of the equation and leading to a change in $\mathbf{E}$, "working" is the component lying in a plane perpendicular to vector $\mathbf{k}$. Therefore, for resolution of the problem, it would have been possible to multiply the fundamental equation vectorly by $-[\mathbf{k}_0[\mathbf{k}_0 \ldots]]$. The result of the action of this operator on rot rot $\mathbf{E}$ is $-\nabla^2\mathbf{E} +$ small terms, and the order of their smallness is higher than that which is considered with subsequent discussion. Thereby the aforementioned replacement is justified, since multiplication by $-[\mathbf{k}_0[\mathbf{k}_0 \ldots]]$ will be performed further; however, the reader will lead up to this operation by more graphic path.

where the linear part of polarization $\mathbf{P}^{l}$ is connected with field $\mathbf{E}$ through the scalar functional operator $\hat{\varkappa}$:

$$\mathbf{P}^{(л)} = \int_0^\infty \varkappa(t')\mathbf{E}(\mathbf{r}, t-t')\,dt', \qquad (5.28)$$

and forced anisotropy is determined only the modulating field $\mathbf{E}_m(\mathbf{r}, t)$. Subsequently, we consider that the right side of (5.27) is small and has the order μ.

With the propagation of light in an anisotropic nonlinear crystal the modulating field practically cannot change polarization of the natural wave. This occurs because the directions of natural polarizations with the assigned direction of the beam vector are determined by optical properties of the crystal directly connected with its spatial symmetry. In an isotropic crystal in the absence of a modulating field, all directions of polarization are natural, the case of degeneration takes place. Superposition of the modulating field on the crystal removes this degeneration – the light wave with propagation over such crystal changes, in general, its polarization.

Thus, the field of the light wave is described by the expression:

$$\mathbf{E} = \mathbf{E}_0(\mu\mathbf{r}, \mu t)\, e^{i(\omega t - \mathbf{kr})} + \mu\mathbf{U}(\mathbf{r}, \mu t)\, e^{i\omega t}, \qquad (5.29)$$

where in contrast to the case of the anisotropic crystal the amplitude $\mathbf{E}_0(\mu\mathbf{r}, \mu t)$ is a vector slowly variable in magnitude and direction but remaining perpendicular to the vector $\mathbf{k}$

$$(\mathbf{kE}_0) = 0. \qquad (5.30)$$

The term $\mu\mathbf{U}(\mathbf{r}, \mu t)$ considers the inaccuracy and is small at all values of coordinates and time. Let us derive the equation describing the behavior of amplitude $\mathbf{E}_0(\mu\mathbf{r}, \mu t)$.

Proceeding just as in the second chapter in the derivation of truncated equations for the amplitdue of the wave in an anisotropic nonlinear dielectric, we will write out the approximate expression

for $\mathbf{P}^{(s)}$. It has the form:

$$\mathbf{P}^{(s)} = \left\{\left[\varkappa(\omega)\mathbf{E}_0 - i\mu\frac{\partial\varkappa}{\partial\omega}\cdot\frac{\partial\mathbf{E}_0}{\partial t}\right]e^{-i\mathbf{kr}} + \mu\varkappa(\omega)\mathbf{U}\right\}e^{i\omega t}. \quad (5.31)$$

The second derivative of this vector with respect to time, which enters into equation (5.27), with the same degree of accuracy is equal to

$$\frac{\partial^2\mathbf{P}^{(s)}}{\partial t^2} = -\omega^2\mathbf{P}^s + 2i\mu\omega\varkappa\frac{\partial\mathbf{E}_0}{\partial t}e^{i(\omega t - \mathbf{kr})}. \quad (5.32)$$

Expression $\frac{\partial^2\mathbf{E}}{\partial t^2}$ is approximately equal to

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = \left[-\omega^2\mathbf{E}_0 e^{-i\mathbf{kr}} + 2i\mu\omega\frac{\partial\mathbf{E}_0}{\partial t}e^{-i\mathbf{kr}} - \mu\omega^2\mathbf{U}\right]e^{i\omega t}, \quad (5.33)$$

and expression $\nabla^2 E$

$$\nabla^2\mathbf{E} = \left[-2i\mu(\mathbf{k}\nabla)\mathbf{E}_0 e^{-i\mathbf{kr}} - k^2\mathbf{E}_0 e^{-i\mathbf{kr}} + \nabla^2\mu\mathbf{U}\right]e^{i\omega t}. \quad (5.34)$$

Substituting all these expressions into (5.27) and considering that

$$k^2 = \frac{\omega^2}{c^2}\left[1 + 4\pi\varkappa(\omega)\right], \quad (5.35)$$

we have:

$$\nabla^2\mathbf{U} + \frac{\omega^2}{c^2}(1 + 4\pi\varkappa)\mathbf{U} = 2i\left\{(\mathbf{k}\nabla)\mathbf{E}_0 + \left[\frac{\omega}{c^2}(1 + 4\pi\varkappa) + \frac{2\pi\omega^2}{c^2}\frac{d\varkappa}{d\omega}\right]\frac{\partial\mathbf{E}_0}{\partial t} + \frac{2\pi i\omega^2}{c^2}\hat{\chi}\mathbf{E}_m\mathbf{E}_0\right\}e^{-i\mathbf{kr}}. \quad (5.36)$$

The linear differential operator, which acts on the vector function $\mathbf{U}$, in the left side of (5.36) has the eigenvalue k and eigenvector — any vector perpendicular to $\mathbf{k}$. Therefore, the right side (5.36) is resonance for the differential operator. For limitedness $\mathbf{U}$ at all values of coordinates and time (requirement of smallness of the correction term), it is necessary that the projection of the vector standing on the right on a plane perpendicular to $\mathbf{k}$, be equal to

zero. This condition is given by the equation describing the behavior $\mathbf{E}_0$. Before writing it out we will consider that according to (5.35)

$$k\frac{dk}{d\omega} = \frac{k}{v_{\text{гр}}} = \frac{1}{c^2}\frac{d}{d\omega}\omega^2(1+4\pi x). \tag{5.37}$$

Replacing the coefficient with a time derivative in the right side of (5.36), in accordance with (5.37) we have:

$$(\mathbf{k}_0\nabla)\mathbf{E}_0 + \frac{1}{v_{\text{гр}}}\frac{\partial \mathbf{E}_0}{\partial t} = +\frac{2\pi i\omega^2}{kc^2}\left[\mathbf{k}_0\left[\mathbf{k}_0\hat{\chi}\mathbf{E}_m\mathbf{E}_0\right]\right]. \tag{5.38}$$

The right side in (5.38) is a projection of vector $\hat{\chi}\mathbf{E}_m\mathbf{E}_0$ on a plane perpendicular to $\mathbf{k}$, and $\mathbf{k}_0$ is a unit vector in the direction of $\mathbf{k}$. The truncated equation (5.38) is an equation describing in a parametric approximation the process of propagation of a light wave in an isotropic nonlinear medium.

The direction of the change in amplitude $\mathbf{E}_0$ is determined, in general, by conditions of the problem. If, as in § 2, the modulating field has the form of a plane traveling or standing wave, then the direction of the change in $\mathbf{E}_0$ coincides with wave vector $\mathbf{k}_m$ of the modulating wave. Designating the coordinate in the direction of this wave vector by ξ, we have $\mathbf{E}_0 = \mathbf{E}_0(\xi, t)$ and

$$\cos\widehat{\mathbf{k}_0\mathbf{k}_m}\frac{\partial \mathbf{E}_0}{\partial \xi} + \frac{1}{v_{\text{гр}}}\frac{\partial \mathbf{E}_0}{\partial t} = +\frac{2\pi i\omega^2}{c^2}\left[\mathbf{k}_0\left[\mathbf{k}_0\hat{\chi}\mathbf{E}_m\mathbf{E}_0\right]\right]. \tag{5.39}$$

The vector operator $\left[\mathbf{k}_0\left[\mathbf{k}_0\hat{\chi}\mathbf{E}_m\ldots\right]\right]$, which acts on vector $\mathbf{E}_0$ in the right side (5.39), turns it in a plane perpendicular to $\mathbf{k}_0$ at a definite angle. This leads, in general, to a change in the plane of polarization of the light wave with its propagation. There are, however, two directions of polarization – eigenvectors of the operator $\left[\mathbf{k}_0\left[\mathbf{k}_0\hat{\chi}\mathbf{E}_m\ldots\right]\right]$, which are not changed in direction during propagation. These directions are determined by properties of symmetry of crystals and by the modulating field. Let us assume that, for example, the modulating field is directed along one of the edges of a cubic crystal (z axis). Then, as follows from the structure of tensor $\hat{\chi}$,

for cubic crystals (Chapter I)

$$P_x = aE_m E_{oy}; \quad P_y = aE_m E_{ox}, \tag{5.40}$$

where $a = \chi_{x,yz} = \chi_{y,xz}$, and the x and y axes are directed along two other edges of the crystal. Let us consider two cases when the light wave propagates along the z axis and when it propagates in the plane xy.

If the wave propagates along the z axis, then in the xy plane there exist two natural directions of polarization $\mathbf{e}_1 = [1,1,0]$ and $\mathbf{e}_2 = [1,-1,0]$. Along these directions $\mathbf{e}_1 \parallel [\mathbf{k}_0[\mathbf{k}_0 \hat{\chi} \mathbf{E}_m \mathbf{e}_1]]$ and $\mathbf{e}_2 \parallel [\mathbf{k}_0[\mathbf{k}_0 \hat{\chi} \mathbf{E}_m \mathbf{e}_2]]$. With entry of a light wave with any polarization into such a crystal along the z axis, the wave is split into two components with polarizations along directions $\mathbf{e}_1$ and $\mathbf{e}_2$. Each of these waves propagates independently of the other with its own phase speed. If $\mathbf{E}_0$ is directed along the x axis, then amplitudes of waves $\mathbf{E}_1$ and $\mathbf{E}_2$ are equal so that $|\mathbf{E}_1| = |\mathbf{E}_2| = \frac{|\mathbf{E}_0|}{\sqrt{2}}$. If the modulating field has the form of a traveling wave with a longitudinal component E_z, and the condition of synchronism (5.11) is fulfilled, then waves $\mathbf{E}_1$ and $\mathbf{E}_2$ undergo phase modulation according to the law studied in § 2.

$$\left.\begin{aligned} \varphi_1 &= \varphi_0 - Bz\cos(\omega_m t - k_m z); \\ \varphi_2 &= \varphi_0 + Bz\cos(\omega_m t - k_m z) \end{aligned}\right\} \tag{5.41}$$

For each of the waves we obtain the final expression:

$$\left.\begin{aligned} E_1 &= \frac{E_0}{\sqrt{2}}\cos\{\omega t - kz + \varphi_0 - Bz\cos(\omega_m t - k_m z)\}; \\ E_2 &= \frac{E_0}{\sqrt{2}}\cos\{\omega t - kz + \varphi_0 + Bz\cos(\omega_m t - k_m z)\}. \end{aligned}\right\} \tag{5.42}$$

In a certain fixed section z_0 there occurs the addition of two mutually perpendicular oscillations with equal amplitudes. The form of the closed curve described by the end of vector $\mathbf{E}$ depends on the difference of phases $\Delta\varphi$ of these oscillations, which, as one can see from (5.42), is equal to:

$$\Delta\varphi = 2Bz\cos(\omega_m t - k_m z_0). \tag{5.43}$$

At a certain instant t the difference of the phases between oscillations is equal to $\Delta\varphi$. Let us see what will occur in section z_0 in the interval of time Δt such that $\cos\omega_m t$ after this time interval almost does not change and $\cos\omega t$ succeeds in accomplishing many oscillations. The given assumption with respect to the time interval Δt means that the difference in phases $\Delta\varphi$ in this interval can be considered constant. Quantity $\Delta\varphi$ determines the form of the ellipse, which is described by the end of vector **E**. Principal axes of the ellipse coincide with axes x and y. In the case $\Delta\varphi=0$ the ellipse degenerates into a straight line. When $\Delta\varphi=\frac{\pi}{2}$ the ellipse is turned into a circumference. The eccentricity of the ellipse changes with a change in $\Delta\varphi$, i.e., with a change in the modulating field. Setting on the path of light the Nicol prism, the amplitude modulation of the light wave can be obtained. If in the initial light wave vector **E** is oriented not along the x axis but in a certain arbitrary direction, then amplitudes of waves on which the initial wave disintegrates prove to be different. With the help of the Nicol prism here it also is possible to carry out amplitude modulation; however, at a great difference in amplitudes of waves the modulation will be less effective, since in this case the eccentricity of the ellipse changes in small limits.

If the modulating field $\mathbf{E}_m$ is spa lly uniform, then for the modulation of the phases we obtain the expression:

$$\varphi_{1,2}=\varphi_0\mp B\frac{1}{\omega_m/2v_{\text{rp}}}\sin\frac{\omega_m z}{2v_{\text{rp}}}\cos\left(\omega_m t-\frac{\omega_m z}{2v_{\text{rp}}}\right). \quad (5.44)$$

and the difference in phases in this case

$$\Delta\varphi=2B\frac{1}{\omega_m/2v_{\text{rp}}}\sin\frac{\omega_m z}{2v_{\text{rp}}}\cdot\cos\left(\omega_m t-\frac{\omega_m}{2v_{\text{rp}}}z\right). \quad (5.45)$$

Let us now examine the second case when the wave vector **k** is located in the xy plane. In this case the light wave with amplitude $\mathbf{E}_0$ breaks up into a wave with polarization along the z axis and a wave with vector of polarization located in the xy plane. The first of these waves in accordance with (5.40) is not modulated with passage through the crystal. The degree of modulation of the second wave

depends on the direction of the propagation of light. If the light wave propagates along the x or y axes, then modulation is absent. If, however, it propagates in directions [1, 1.0], [1, -1.0] or opposite directions, then the degree of modulation is maximum.

If the field of modulation has the form of a traveling wave, and the condition of synchronism of it is carried out with the light wave propagating in direction [1, 1.0], the process of modulation is described by expressions:

$$\varphi_1 = \varphi_0$$
$$\varphi_2 = \varphi_0 + B\xi \cos(\omega_m t - k_m \xi), \tag{5.46}$$

where ξ – coordinate in direction $\mathbf{k}_m$, and the difference in phases

$$\Delta\varphi = B\xi \cos(\omega_m t - k_m \xi). \tag{5.47}$$

For the case of a uniform modulating field

$$\Delta\varphi = B \frac{1}{\omega_m/2v_{\text{гр}}} \sin \frac{\omega_m \xi}{2v_{\text{гр}}} \cos\left(\omega_m t - \frac{\omega_m}{2v_{\text{гр}}} \xi\right). \tag{5.48}$$

In conclusion one should note that the second method of amplitude modulation, at which the ray of light is perpendicular to the modulating field, at equal intensities of this field and at equal amplitudes of components of waves is twice less effective than the method at which the light ray coincides in direction with the field strength of modulation. However, design advantages of the second method are quite great, and in a number of systems of modulation its use is preferable (see [195]).

§ 5. Conversion of the Form of Modulation with Parametric Amplification of Traveling Waves

As was already indicated in the introduction to this chapter, an important class of problems on modulated waves in nonlinear media are problems on the transformation of the form of modulation of the wave. Actually, in a highly dispersive medium there can be created

conditions at which even the coherent generating of the second harmonic is impossible. A harmonic wave of frequency ω_0 in such a medium will propagate in exactly the same way as in the linear case (the dispersion already in frequency band $\omega_0, 2\omega_0$ is quite great). Another situation takes place if the wave is modulated; the presence of nonlinearity can lead to stored distortions of the form of modulation of the wave, although the wave itself remains here quasi-monochromatic. It is easy to understand the latter if one were to turn to spectral concepts. The modulated wave occupies a finite spectral interval $\Delta\omega$; when $\frac{\Delta\omega}{\omega_0} \ll 1$ the interaction of different components lying in the band $\Delta\omega$ can lead to the appearance of stored effects, inasmuch as dispersion in the band $\Delta\omega$ is expressed weakly. One of the examples of the conversion of modulation in a highly dispersive medium – distortion of amplitude modulation in a nonlinear medium – is examined by Ostrovskiy [173]. An interesting result of the calculation conducted by him is the conclusion concerning the possibility of the appearance of Riemannian waves of enveloping in a highly dispersive medium.

Below we will examine another problem on the conversion of modulation: we will analyze the process of conversion of modulation with the interaction of two waves with multiple frequencies ($\omega_1=\omega$; $\omega_2=2\omega$) in a quadratic medium. We will consider that the field strength of the wave at frequency 2ω considerably exceeds the field strength at frequency ω; therefore, the problem stated can be solved in a parametric approximation (see § 4 of Chapter II). Thus, let us assume that the polarizability of the medium has the form:

$$\hat{\varkappa}(z,t,t') = \hat{\varkappa}(t') + \hat{M}(t')\exp i\,[2\omega t - k_{\text{н}} z]\,. \tag{5.49}$$

The field at frequency ω will be written in the form

$$\mathbf{E} = \mathbf{e}_0 A(\mu t, \mu \mathbf{r}) \exp i\,[\omega t - \mathbf{kr}]\,. \tag{5.50}$$

We will consider, as always, that

$$\mathbf{k}_{\text{н}} = 2\mathbf{k} + \Delta;\ |\Delta|/k \sim \mu\,.$$

Calculating with the help of (5.49) and (5.50) polarization of the medium at frequency ω (see formula (2.103a)), substituting the obtained expressions in the Maxwell equations and using the standard method discussed in Chapter II, we arrive at the truncated equations — equations of the parametric smplifier of the traveling wave:

$$\cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cdot \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{\partial A}{\partial z} + \frac{1}{v_{rp}} \frac{\partial A}{\partial t} + \mathbf{e}_0 \hat{\alpha} \mathbf{e}_0 A + i\eta\omega^2 e^{-2i\Delta r} A^* = 0 \quad (5.51a)$$

$$\cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{\partial A^*}{\partial z} + \frac{1}{v_{rp}} \frac{\partial A^*}{\partial t} + \mathbf{e}_0 \hat{\alpha} \mathbf{e}_0 A - i\eta\omega^2 e^{2i\Delta r} A = 0. \quad (5.51b)$$

Here $\eta = \frac{2\pi}{kc^2} \mathbf{e}_0 \hat{M}(\omega) \mathbf{e}_0$ [compare (2.78)].

From (5.51) it is clear that the behavior of amplitude A is described by a system of two truncated equations (complex) in partial derivatives, which are not split and which must be solved jointly. Here there formally occurs an interaction of waves with frequencies $+\omega$ and $-\omega$. Actually this means that the behavior of waves with various phases with respect to the phase of the wave, change of parameter (5.49) differently (see also § 4 of Chapter IV).

To solve the system of equations (5.51), let us introduce new variables:

$$\xi_1 = \frac{1}{2} (z + t v_{rp} \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0});$$

$$\xi_2 = \frac{1}{2} (z - t v_{rp} \cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0}). \quad (5.52)$$

Equations (5.51) then obtain the form:

$$\cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA}{d\xi_1} + \mathbf{e}_0 \hat{\alpha} \mathbf{e}_0 A + i\eta\omega^2 e^{-2i(\Delta' \xi_1 + \varphi')} A^* = 0$$

$$\cos \widehat{\mathbf{k}_1 \mathbf{s}_1} \cos \widehat{\mathbf{s}_1 \mathbf{z}_0} \frac{dA^*}{d\xi_1} + \mathbf{e}_0 \hat{\alpha} \mathbf{e}_0 A^* - i\eta\omega^2 e^{2i(\Delta' \xi_1 + \varphi')} A = 0, \quad (5.53)$$

where $\Delta' = \Delta_z$, and $\varphi' = \Delta_z \xi_2$. Equations (5.53) are ordinary differential equations with respect to the argument ξ_1, and argument ξ_2 enters as a parameter. They can be solved in general form; however, this solution is very bulky, and we will analyze the equations for the case $\Delta = 0$. Then we have:

$$A = A_1(\xi_2)\exp\left[-\frac{e_0\hat{a}e_0 - \omega^2\eta}{\cos\hat{k_1 s_1}\cos\hat{s_1 z_0}}\xi_1\right] +$$
$$+ A_2(\xi_2)\exp\left[-\frac{e_0\hat{a}e_0 - \omega^2\eta}{\cos\hat{k_1 s_1}\cos\hat{s_1 z_0}}\xi_1\right], \quad (5.54)$$

where on arbitrary complex functions $A_1(\xi_2)$ and $A_2(\xi_2)$ by equations (5.53) there are imposed limitations of the form:

$$A_1 = iA_1^*; \quad A_2 = -iA_2^*. \quad (5.55)$$

This means that

$$A_1(\xi_2) = (1+i)B_1(\xi_2); \quad A_2(\xi_2) = (1-i)B_2(\xi_2), \quad (5.56)$$

where $B_1(\xi_2)$ and $B_2(\xi_2)$ – arbitrary real functions. Finally for A we have:

$$A = B_2(\xi_2)(1+i)\exp\left[-\frac{e_0\hat{a}e_0 - \omega^2\eta}{\cos\hat{k_1 s_1}\cos\hat{s_1 z_0}}\xi_1\right] +$$
$$+ B_2(\xi_2)\cdot(1-i)\exp\left[-\frac{e_0\hat{a}e_0 - \omega^2\eta}{\cos\hat{k_1 s_1}\cos\hat{s_1 z_0}}\xi_1\right]. \quad (5.57)$$

Using (5.57), one can determine the form of modulation in the arbitrary section z according to the assigned modulation at the input the assignment of boundary conditions permits uniquely determining functions $B_1(\xi_2)$, $B_2(\xi_2)$.

In the investigation of distortions of modulated signals in nonlinear media, it is frequently more convenient to use equations for real amplitudes and phases. The equivalent (5.53) equations for the real amplitude A and phase ϕ of the modulated signal in a degenerated parametric amplifier of a traveling wave have the form [compare (4.69)-(4.70)]:

$$\frac{dA}{d\xi_1} + \sigma A_н A \sin 2\varphi + \delta A = 0; \quad (5.58a)$$

$$\frac{d\varphi}{d\xi_1} + \Delta + \sigma A_н \cos 2\varphi = 0. \quad (5.58b)$$

Here designations σ, δ, and $A_{н}$ are analogous to designations accepted in Chapters III-IV.

Truncated equations (5.58) should be solved with boundary conditions set at $z=0$. It is necessary to consider that in new variables ξ_1, ξ_2 the point $z=0$ corresponds to $\xi_1=-\xi_2$ [see (5.52)]. Therefore, if in variables t, z boundary conditions have the form:

$$z=0;\quad A(0,t)=A_0(t);\quad \varphi(0,t)=\varphi_0(t), \tag{5.59}$$

that in variables ξ_1, ξ_2 we have:

$$\xi_1=-\xi_2;\quad A=A_0(-\xi_2);\quad \varphi=\varphi_0(-\xi_2). \tag{5.60}$$

Using (5.58) and (5.60), one can determine, for example, the law of the change in phase of the amplified wave (we will limit ourselves for simplicity, as earlier, to the case $\Delta=0$). Inasmuch as the phase equation can be integrated independently of the amplitude, by conducting integration and passing to variables t, z, we obtain:

$$\varphi(t,z)=\operatorname{arctg}\left[e^{-\frac{\delta A_{н} z}{v_{гр}\cos \widehat{k_1 s_1} \cos \widehat{s_1 z_0}}} \times \right.$$
$$\left. \times \operatorname{tg}\varphi_0\left(t-\frac{z}{v_{гр}\cos \widehat{k_1 s_1}\cos \widehat{s_1 z_0}}\right)\right]. \tag{5.61}$$

From (5.61) it is clear that with a growth in z the index of the phase modulation in the degenerated amplifier of the traveling wave decreases.

Using the result of (5.61), it is possible to integrate the amplitude equation.

The modulation of the amplitude when $z>0$ is determined not only by the amplitude, but also by phase modulation of the input signal.

We will not discuss more specifically the examined example; its detailed analysis is given by us in [56]. Here we will underline only that the change in variables (5.52) is very effective in the theory of modulated waves in nonlinear dispersive media, inasmuch as it allows reducing partial differential equations to equations in common derivatives. In [56] this procedure was used for investigating statistical phenomena in the parametric amplifier of a traveling wave.

The same approach can appear, apparently, expedient in the investigation of problems connected with the generation of harmonics by modulated waves, with the investigation of statistical phenomena with nonlinear wave interactions and so on.

In this chapter we were limited to the examination of modulated waves in a quadratic medium.

Similar problems can be of considerable interest for the cubic medium; with this here certain effects connected with the presence of nonlinear corrections to the dielectric constant are possible. As an example let us indicate that in the cubic medium there can take place the effect of cross-modulation — an amplitude-modulated wave in a cubic medium modulates the phase of a weak wave, etc. (see, for example, § 4 of Chapter III, where there is introduced the concept of "nonlinear detuning"). Let us also note that effects connected with the change in constant polarization of the medium in the field of the modulated wave were examined recently in the work of Askar'yan [203].

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(U) The nonlinear effects examined in this book are effects of the first order with respect to small parameters, parameter of anharmonicity and parameter characterizing the ratio of the shift of the charged particle to the wavelength. The rapid progress in laser technology at present provides the obtaining of such electromagnetic field strengths at which effects of the second order can appear. Nonlinear scattering is an interesting effect of the second order. In further development the theory of waves in a nonlinear medium, discussed in Chapter 2 is needed. Here the primary interest is in the propagation of it on light beams of finite aperature, converging beams etc. An account of the finite width of the spectrum of interacting waves is also very important (discussed in Chapter 6). It should be noted that in a number of cases the real two-dimensional problem is reduced to an equivalent one-dimensional problem by an appropriate selection of the "frequency difference" vector (Chapters 2,3,and 6). It should also be noted that the forced combination (Raman) scattering is not, of course, the only example of nonlinear interaction, where part of the energy of interacting electromagnetic waves gives rise to oscillations of the medium not possessing an electrical dipole moment. In a number of problems on nonlinear effects in crystals, acoustic oscillations ("forced Rayleigh scattering") should be considered.